Splitting Strategies for Structured Illumination Microscopy Reconstruction

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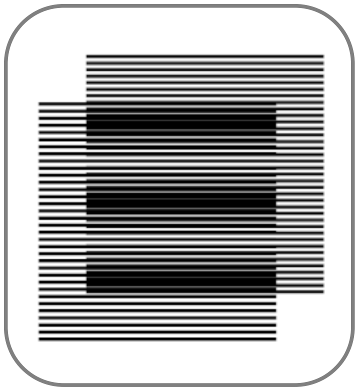
Journées Optimisation Fonctionnelle – CNES, COMET-TSI

Wednesday June 26th 2024, ISAE-Toulosue

Outline of the Talk

SIM Principle



ADMM for SIM

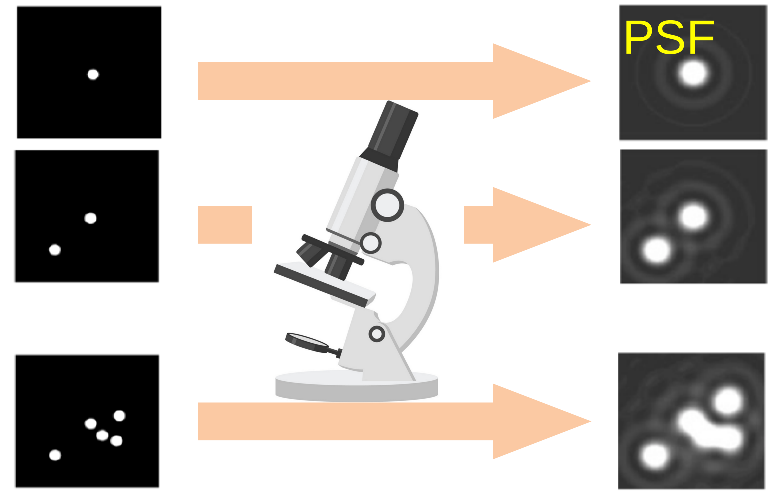
GlobalBioIm

Resolution Limit in Microscopy

Diffraction Phenomenon

- Limited physical aperture of optical components
- Restrict the spatial extent of the wavefront

The system response to a source point is a blurry spot



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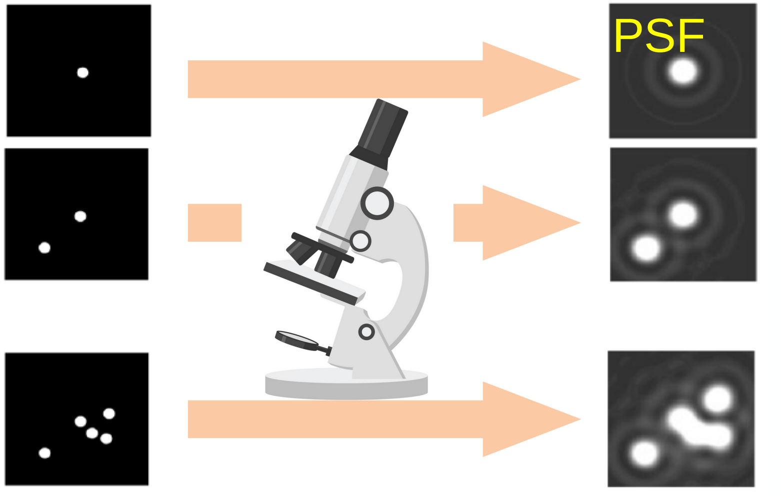

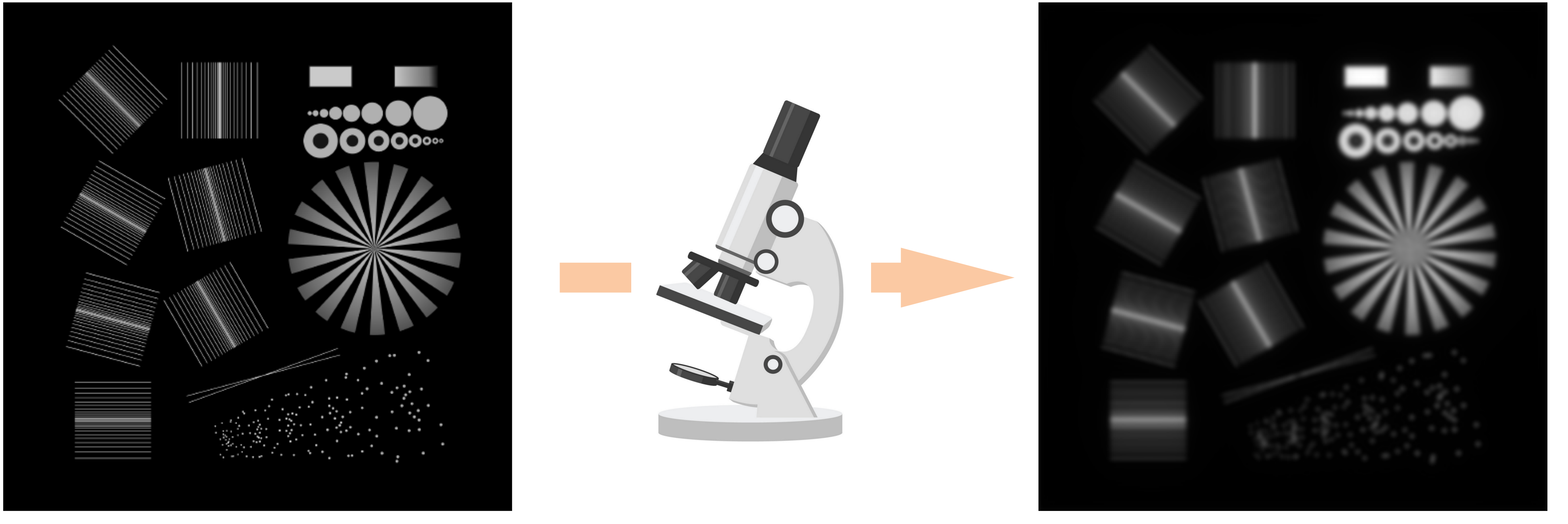

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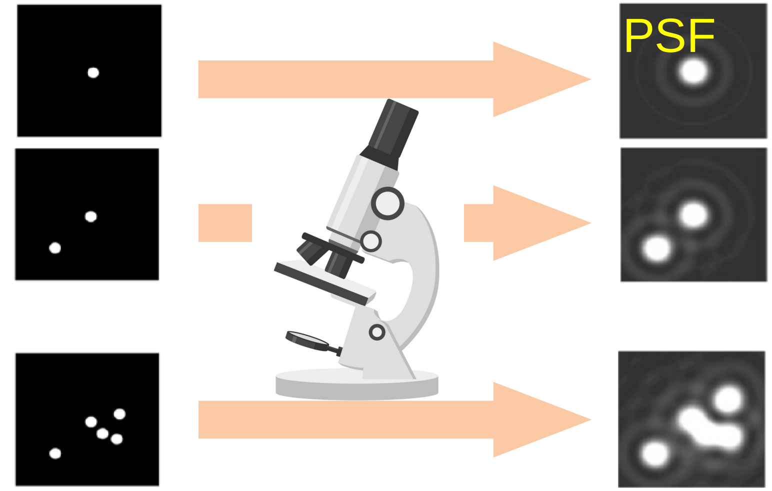

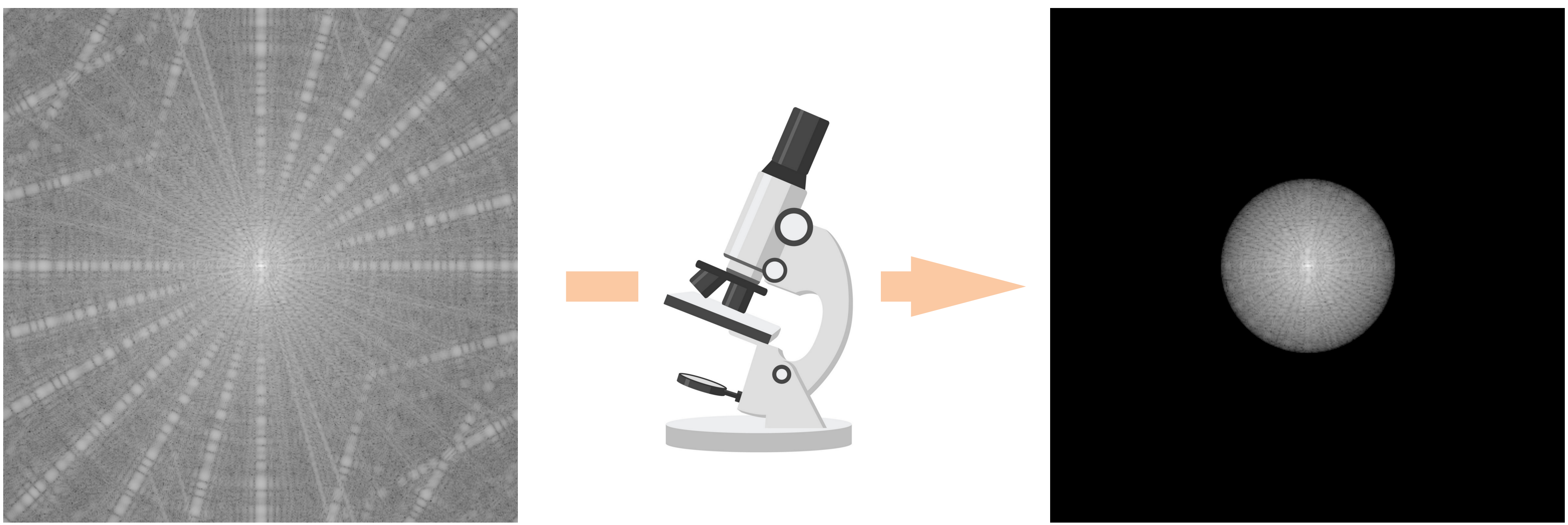

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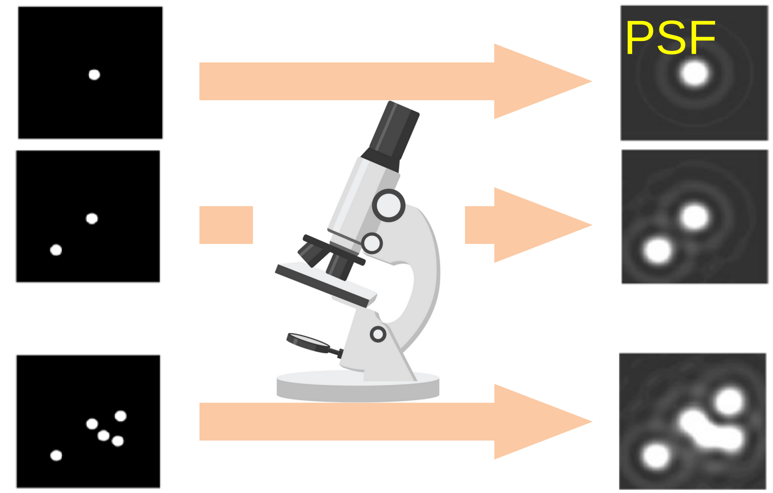
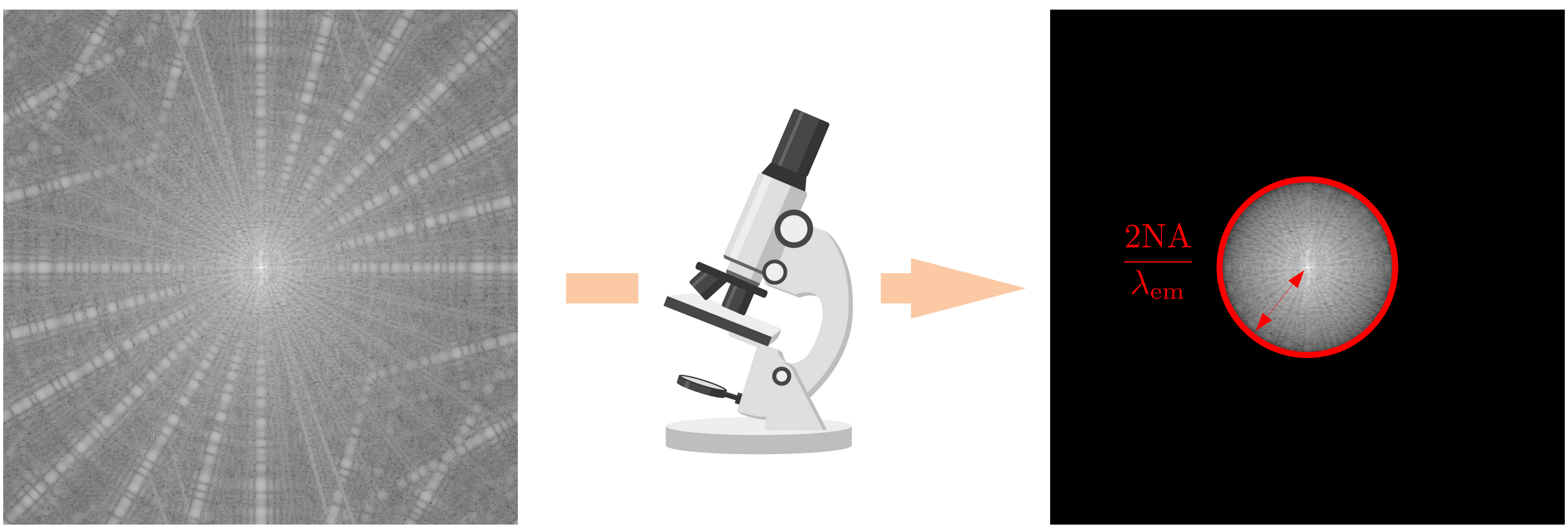
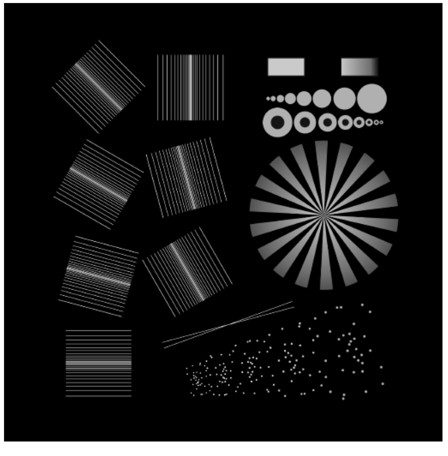


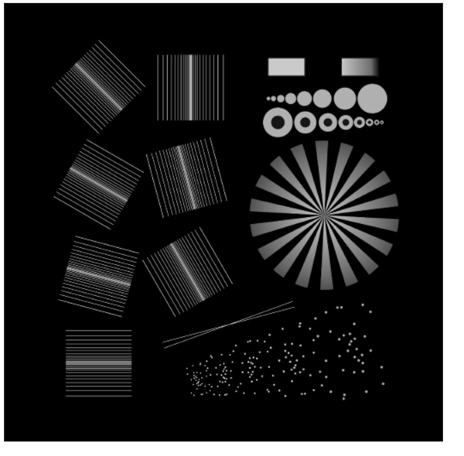


Resolution Limit in Microscopy

Sample



“Ideal” image



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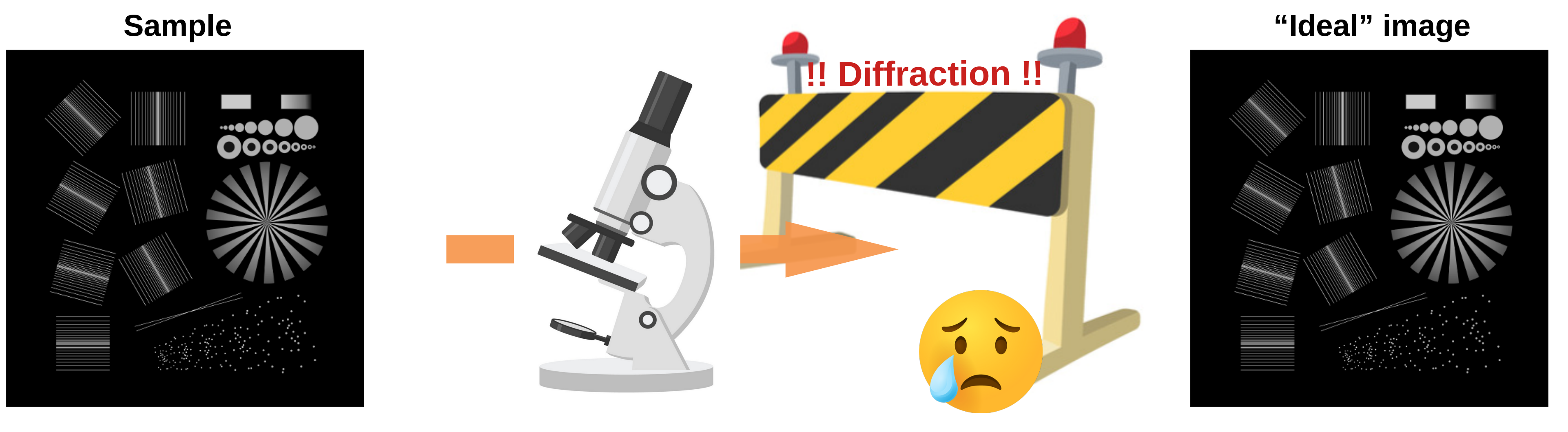

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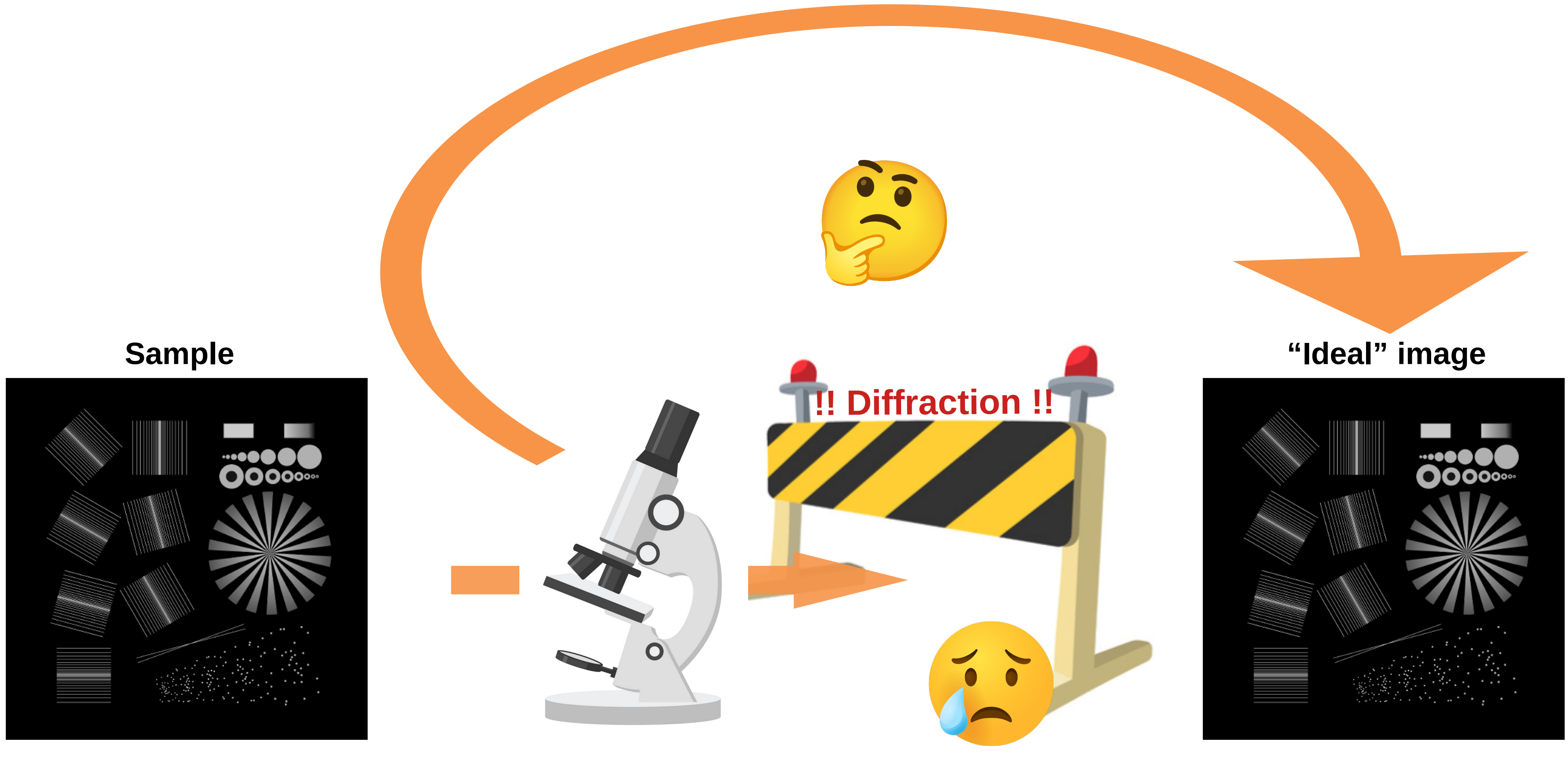
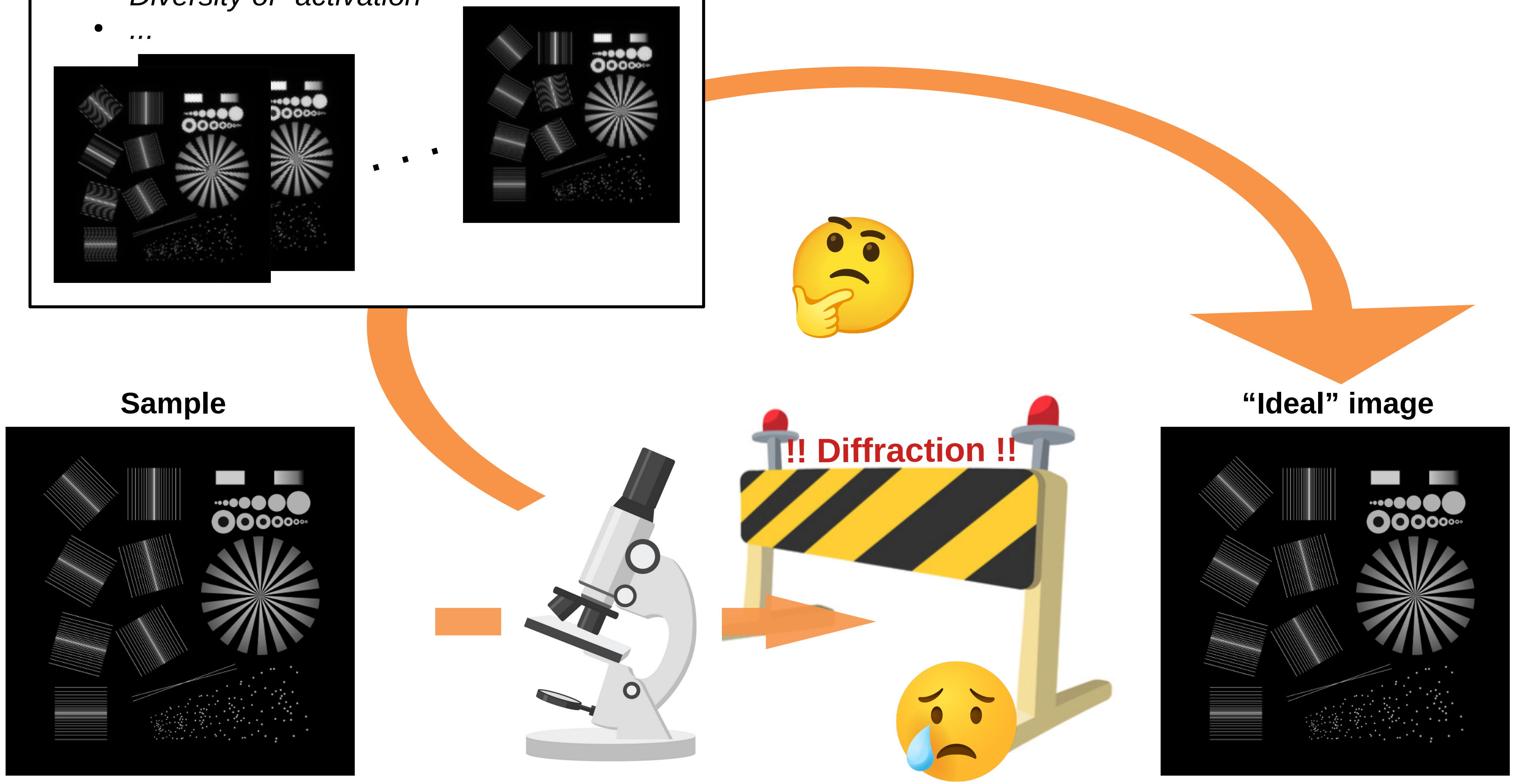


Resolution Limit in Microscopy

1) Acquire a set of diffraction limited images

- *Diversity of illuminations*
- *Diversity of orientations*
- *Diversity of "activation"*
- *...*



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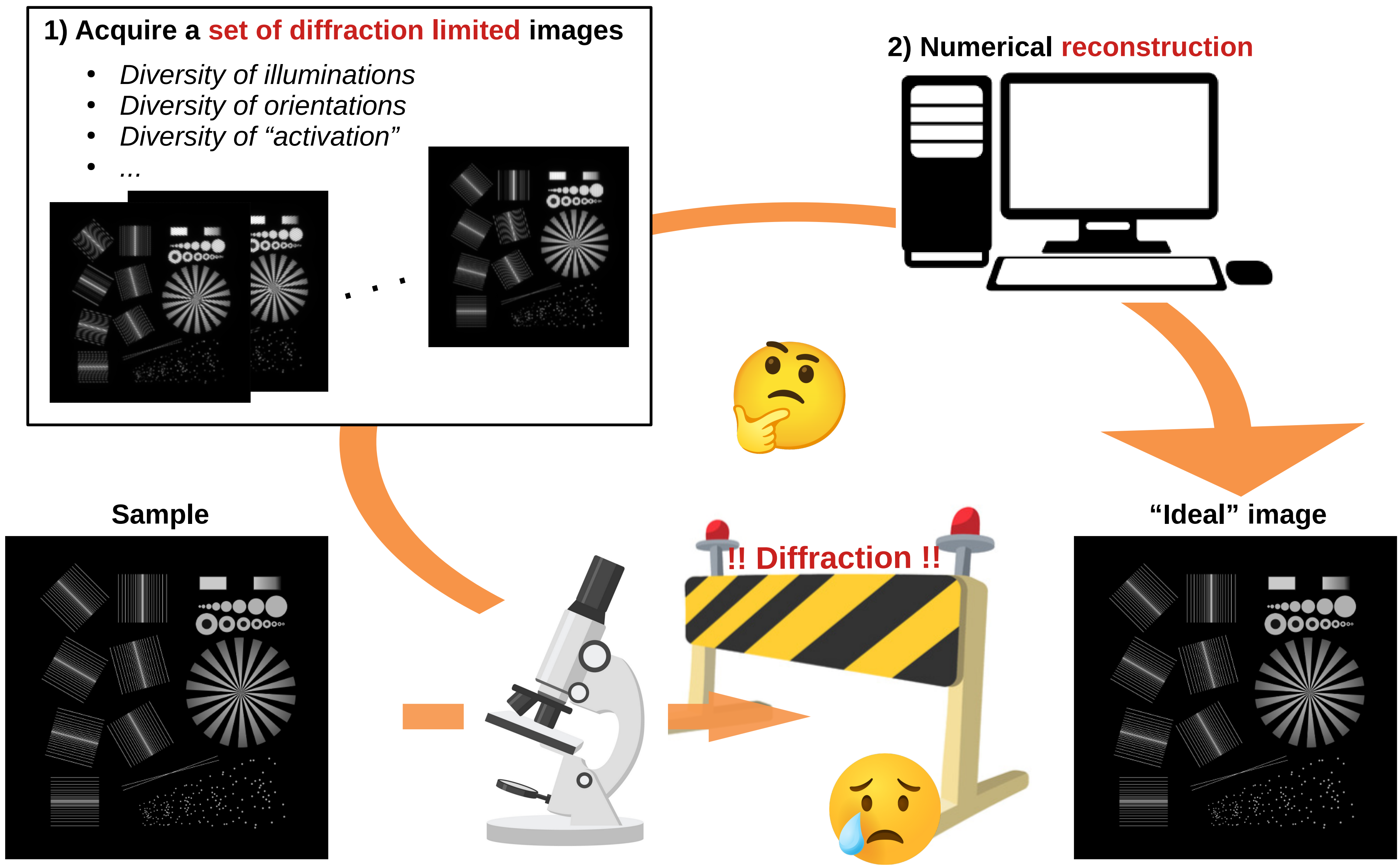
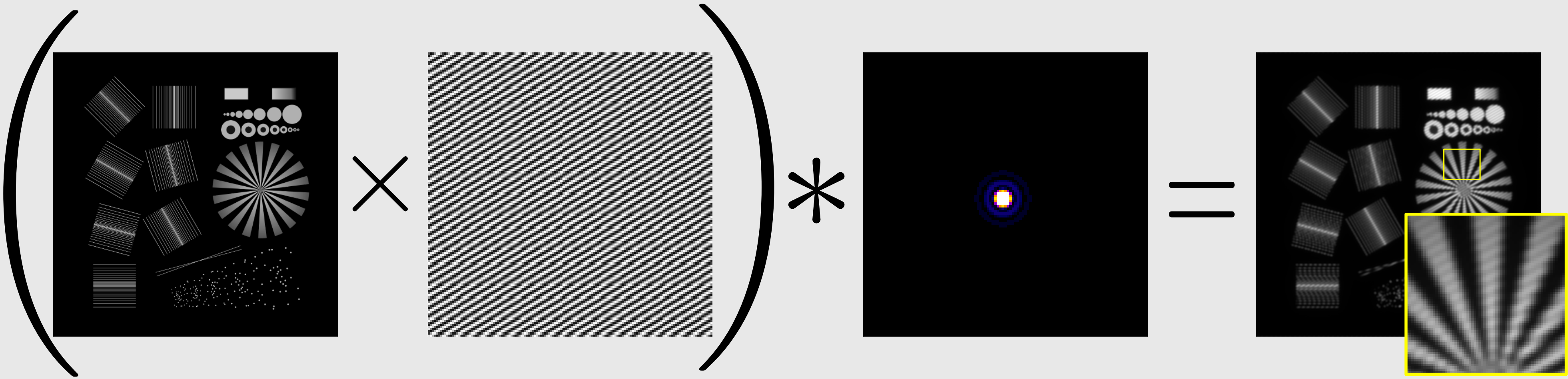


Structured Illumination Microscopy – Image Formation

Modification of the illumination (sinusoidal pattern)



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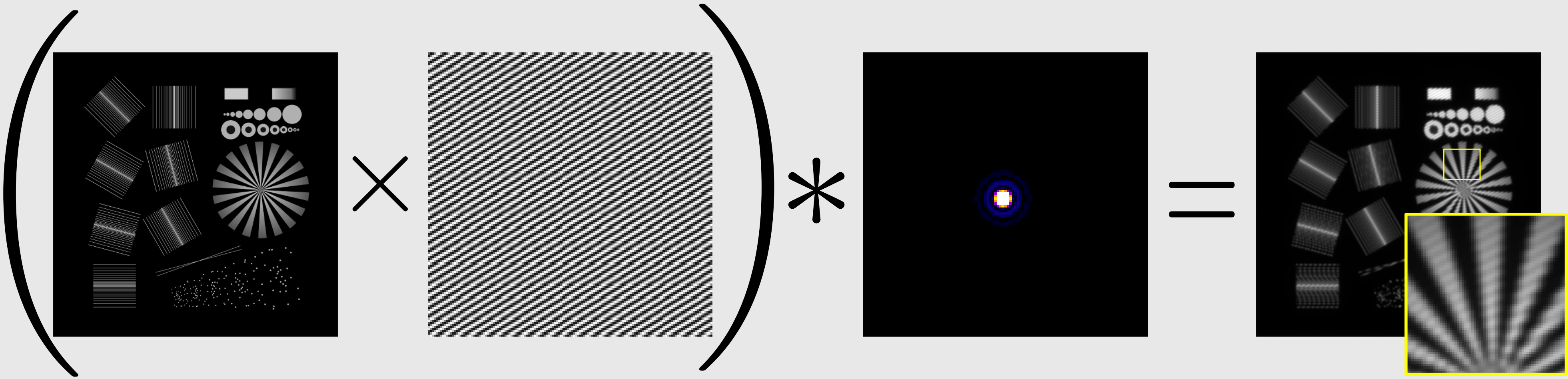

In Fourier domain...

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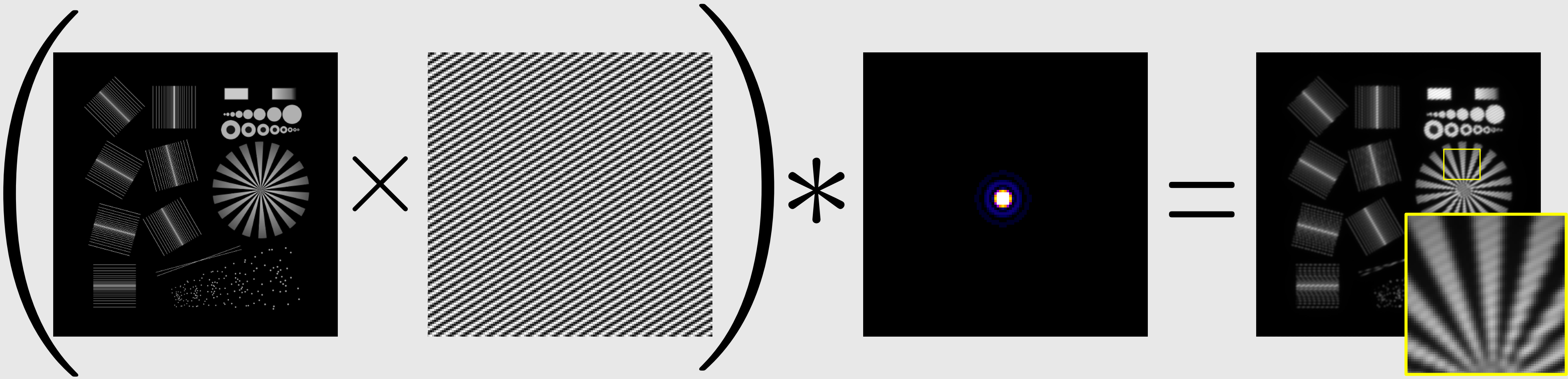

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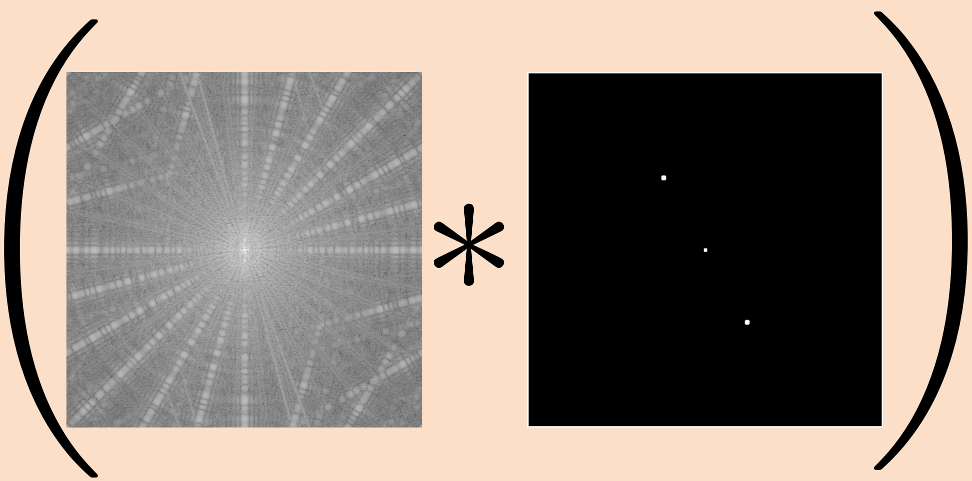

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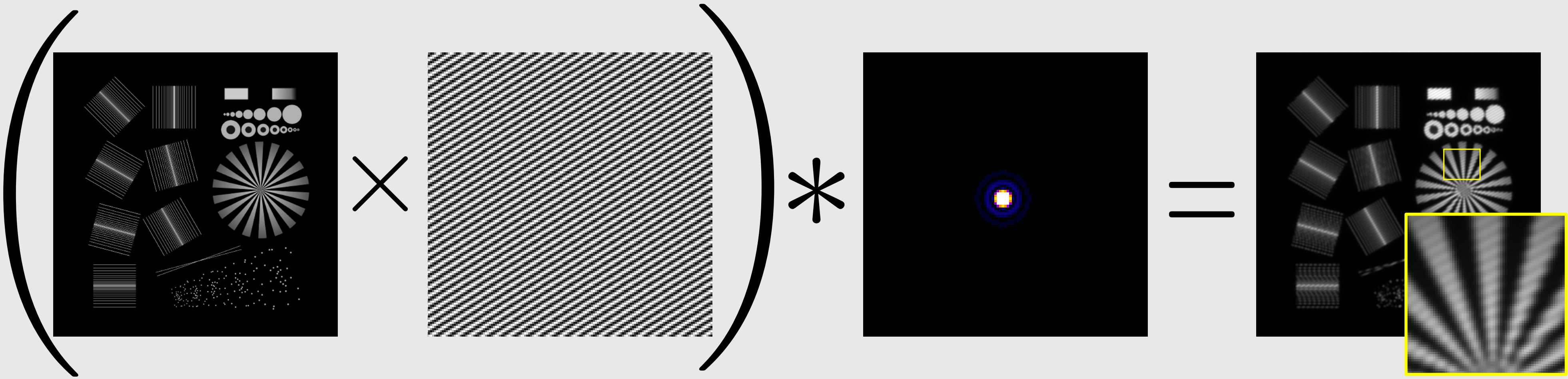

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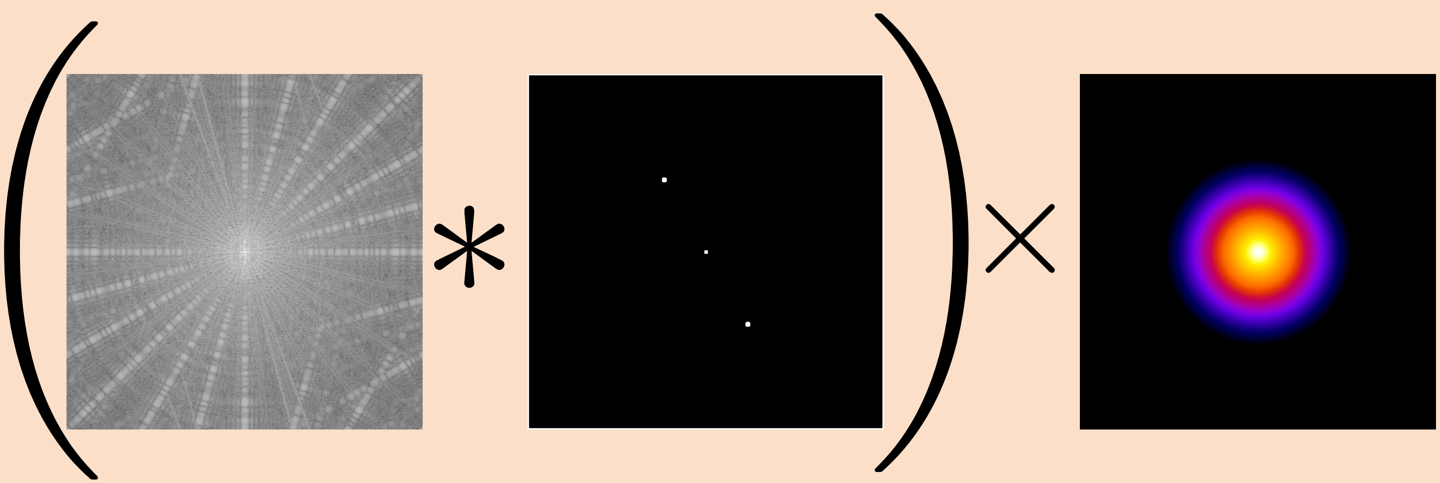

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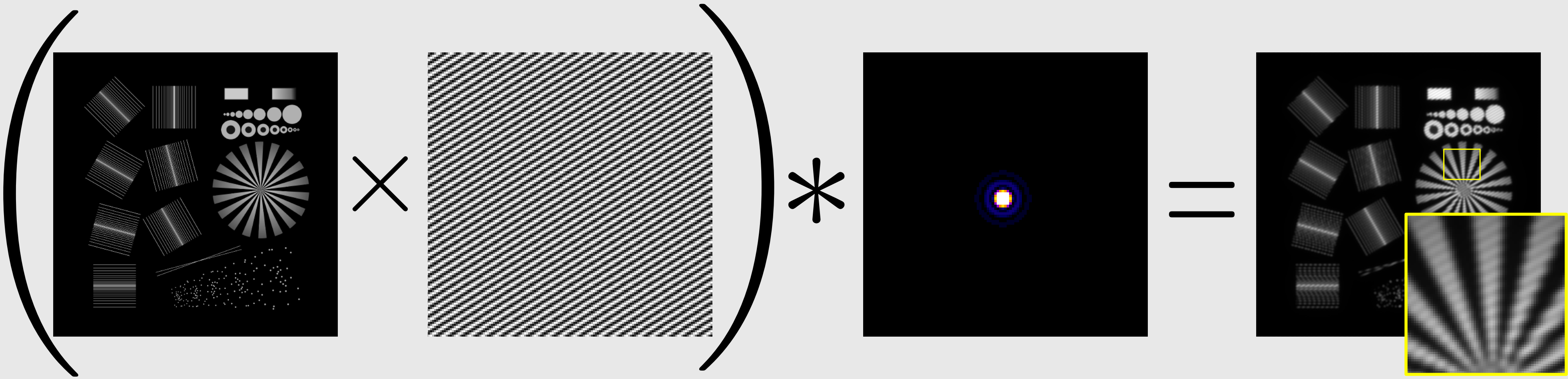

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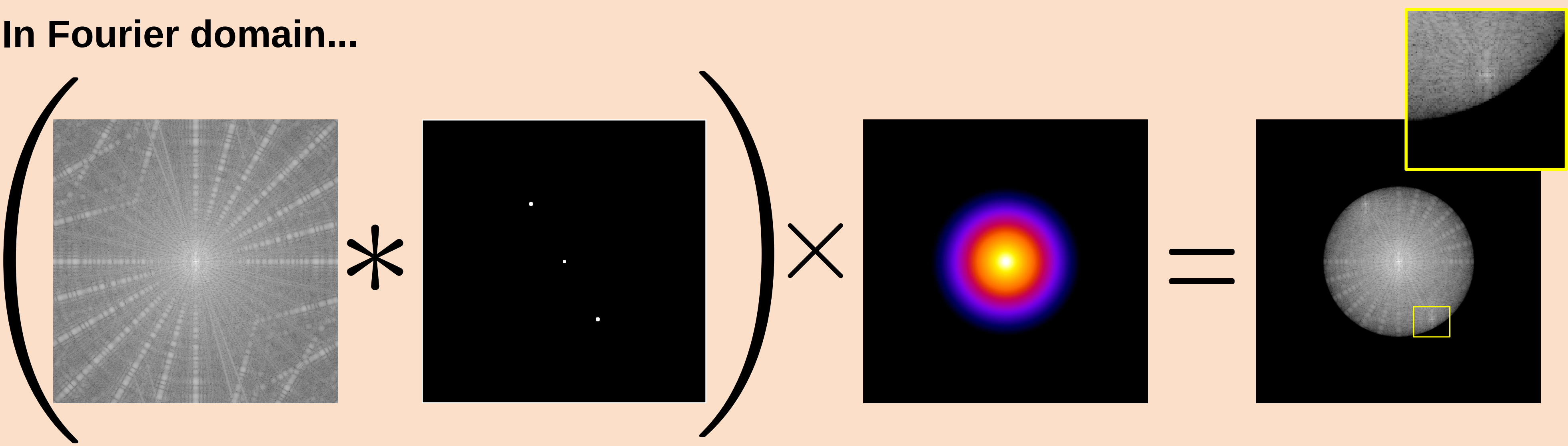

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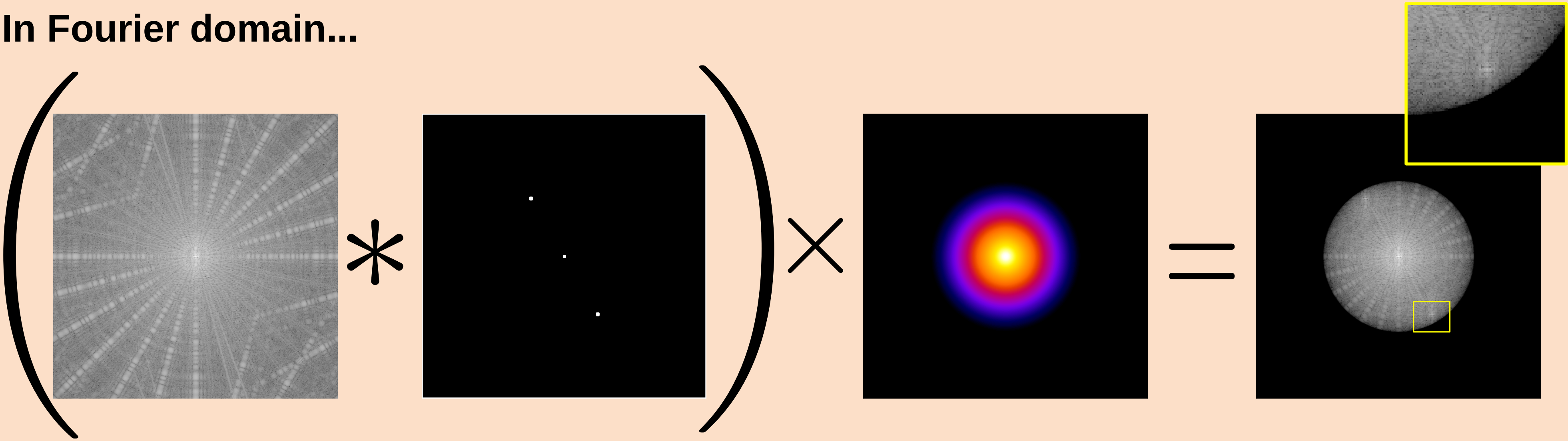

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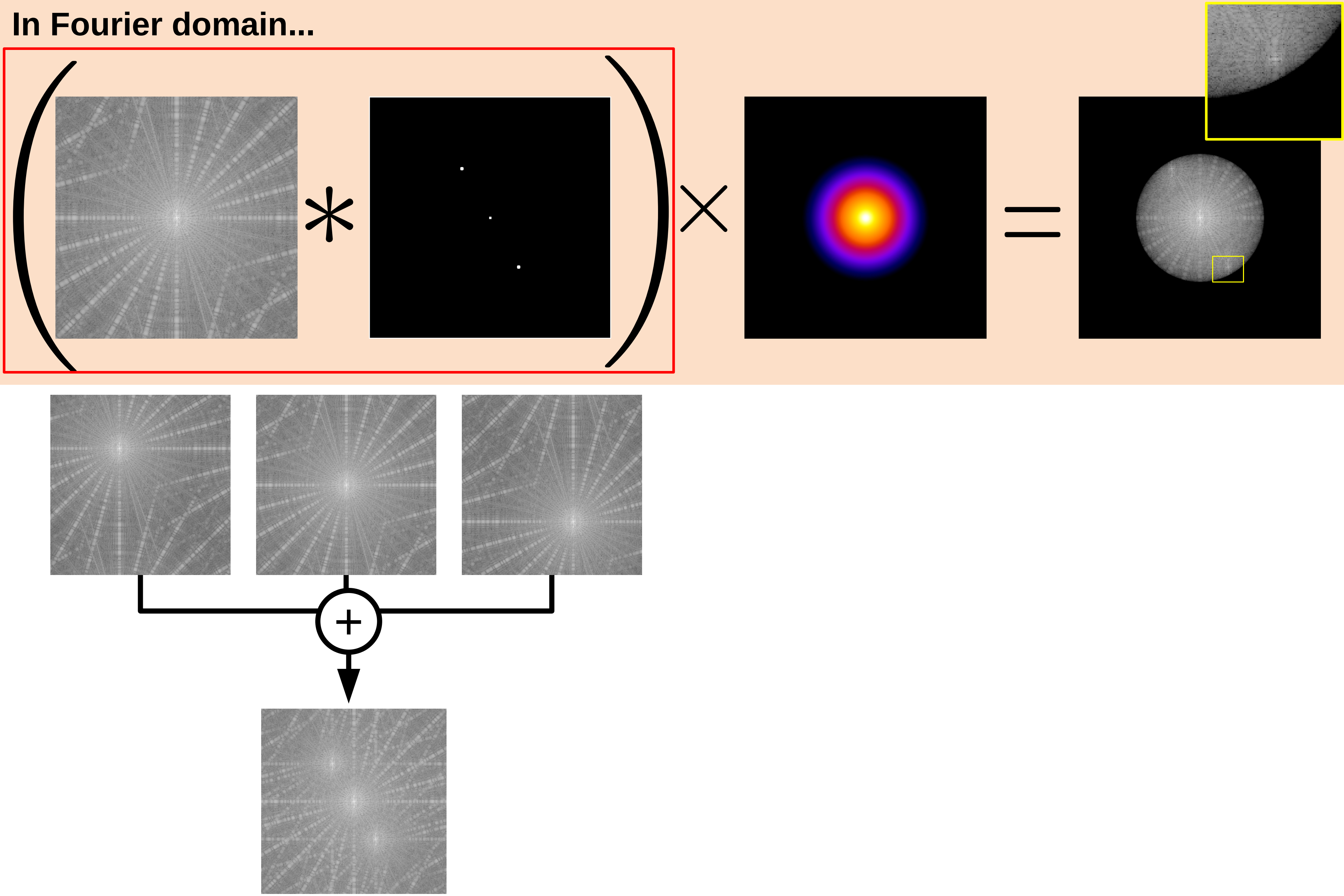

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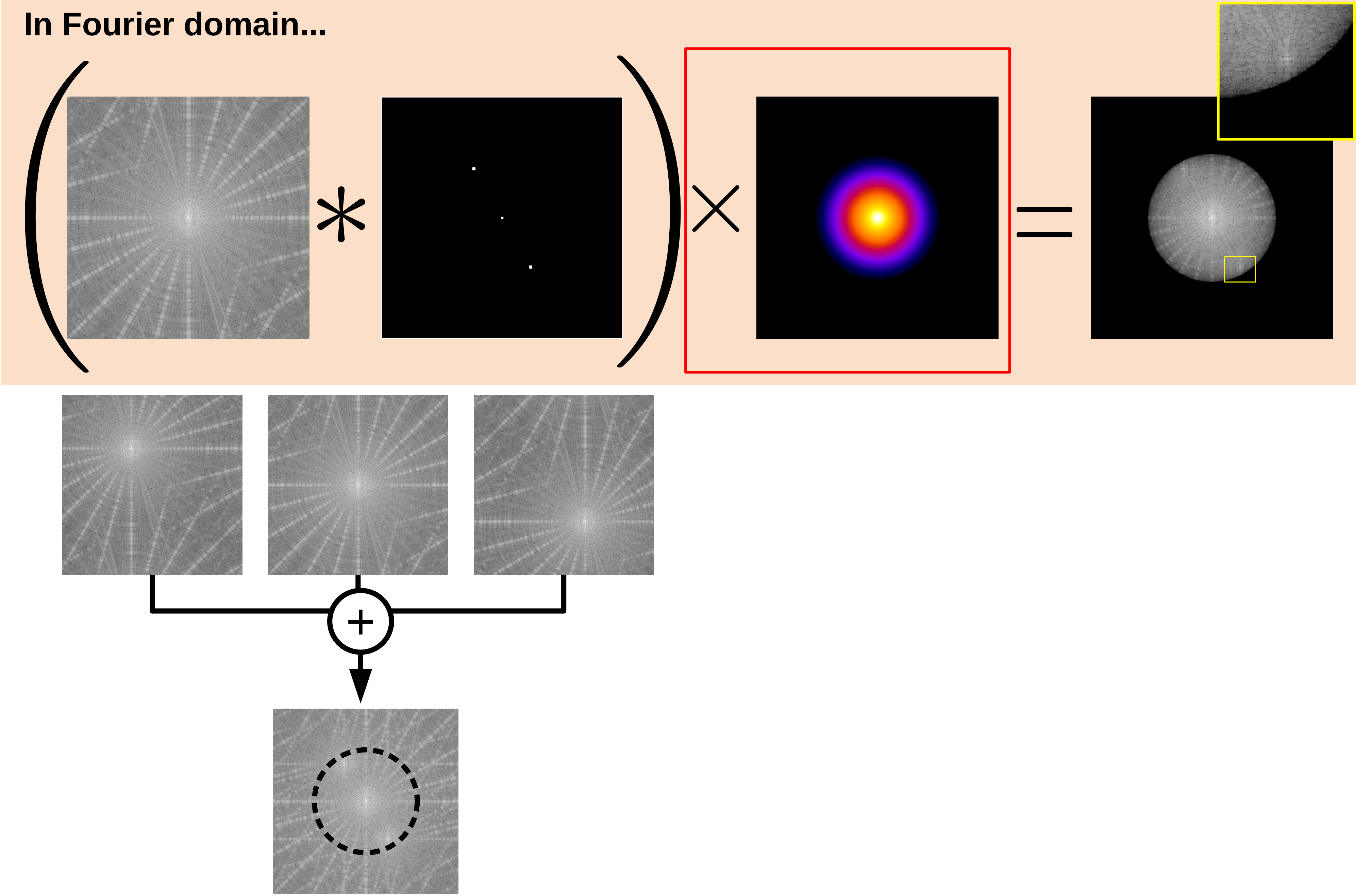

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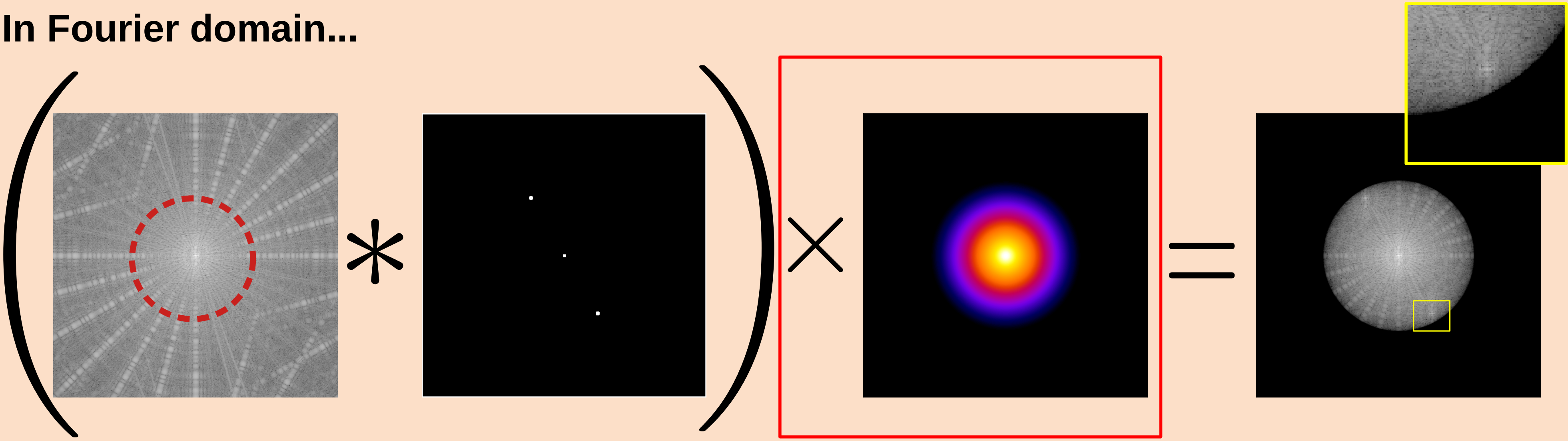

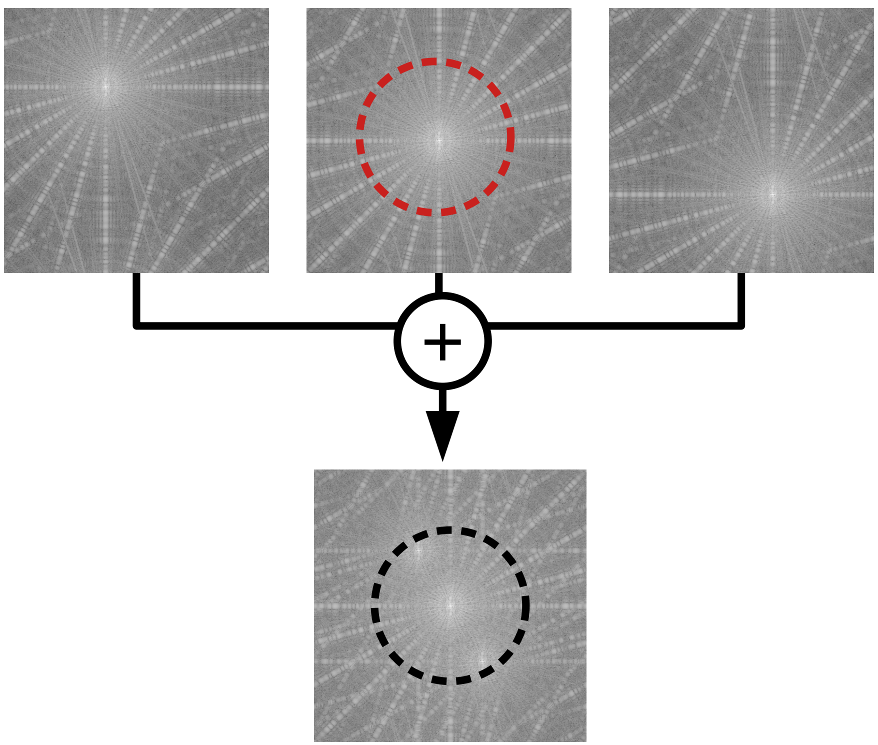

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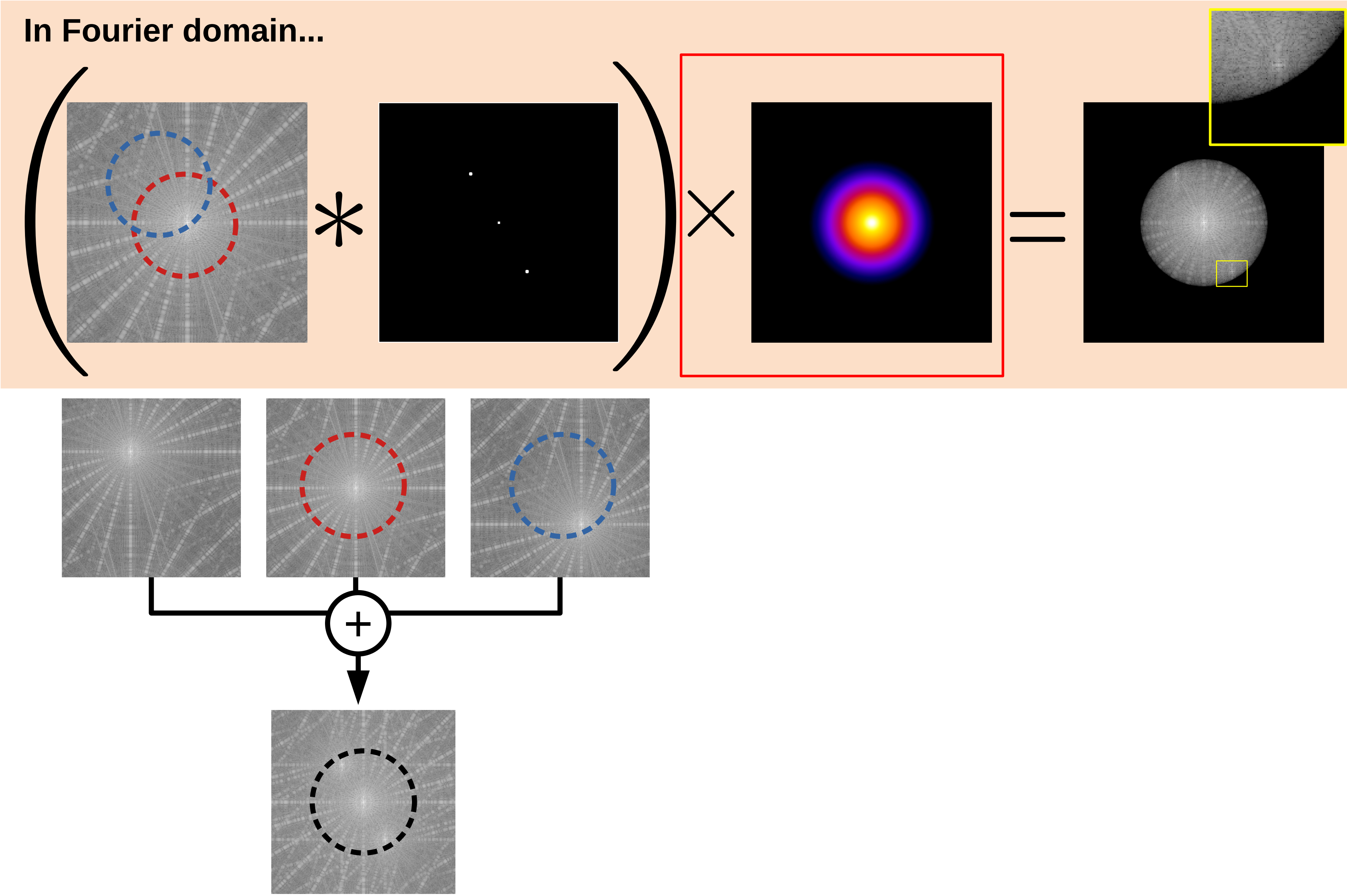

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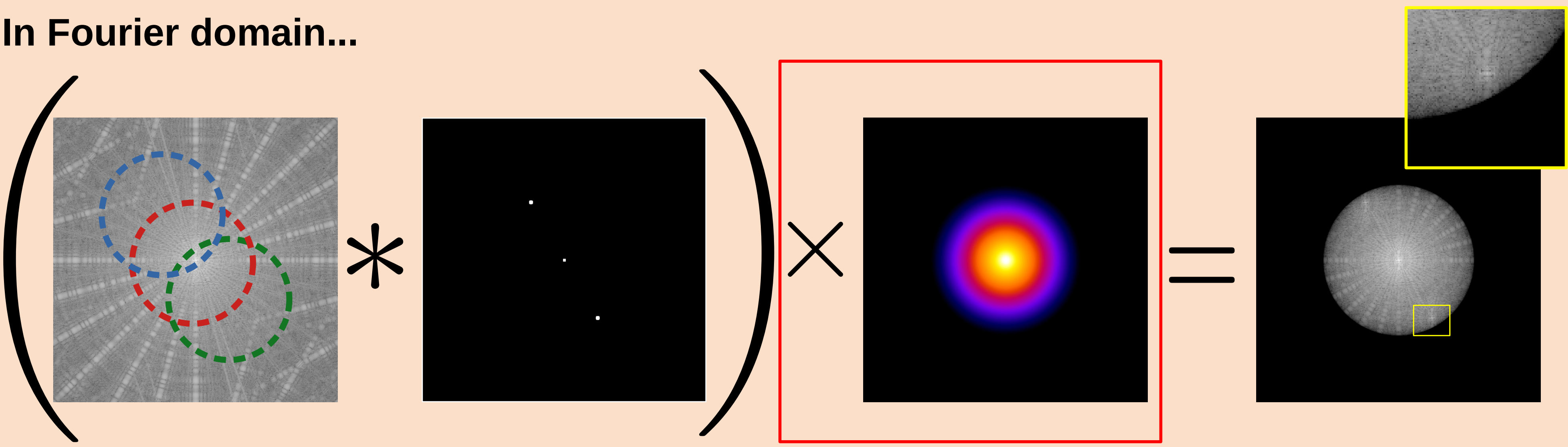

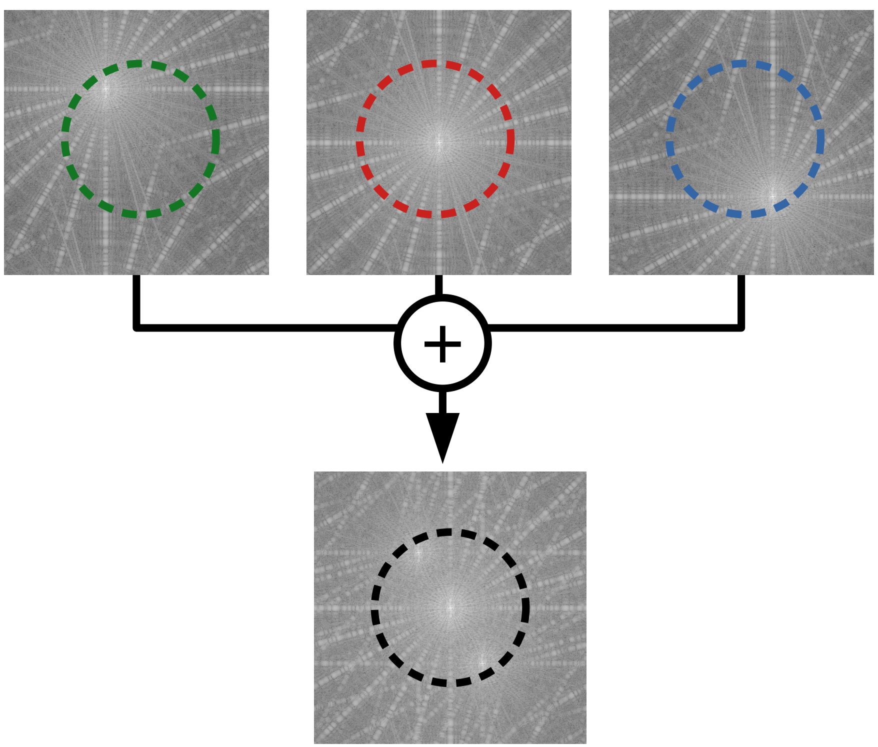

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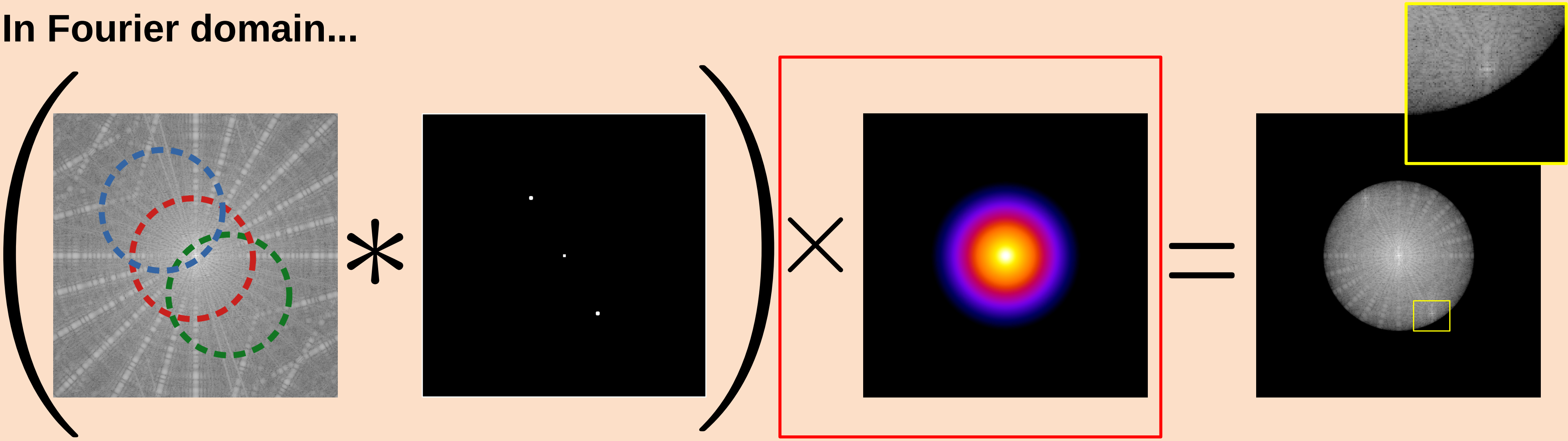
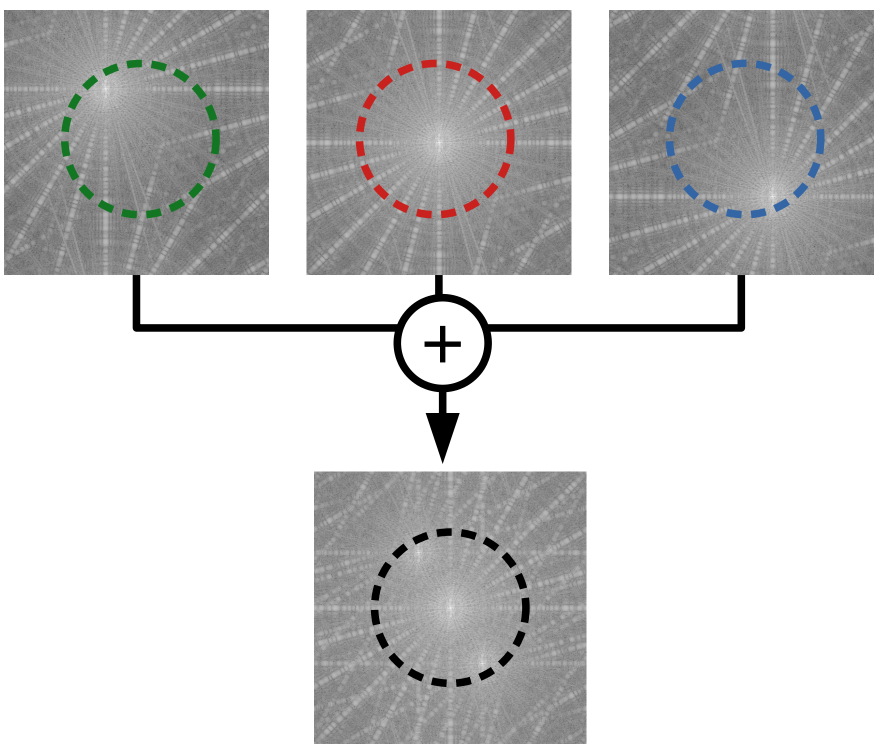
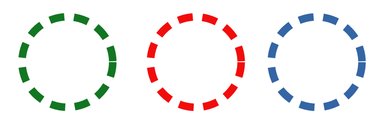




Using structured illuminations, high-frequency components of the sample are shifted into the OTF bandwidth.

Reconstruction is needed to “unmix” the three frequency components

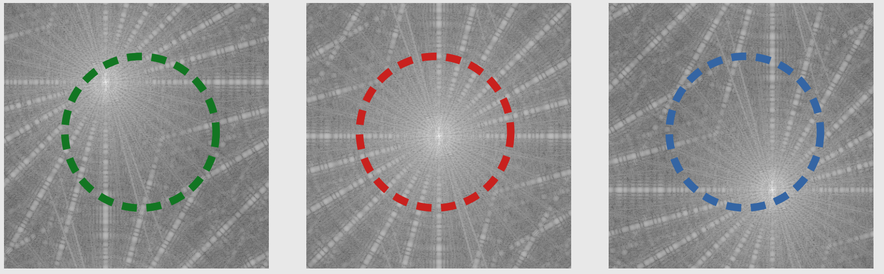


… and shift them back to their correct location in the Fourier plane.

Structured Illumination Microscopy – Reconstruction

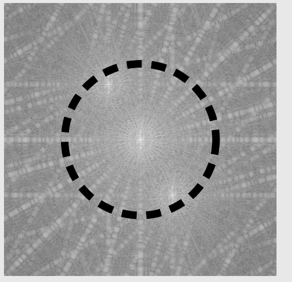
For a single SIM image:

3 unknown components to unmix



with

1 “measure”



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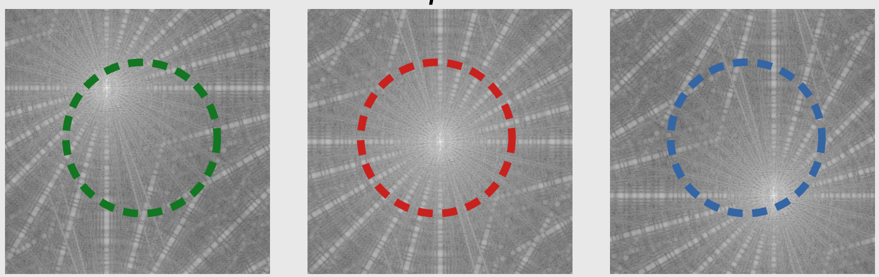

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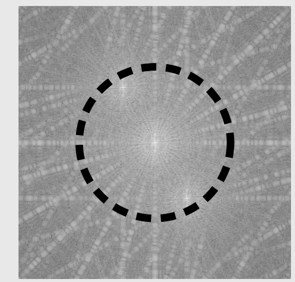

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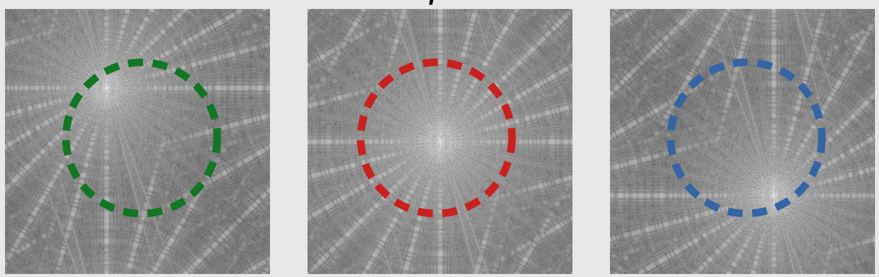

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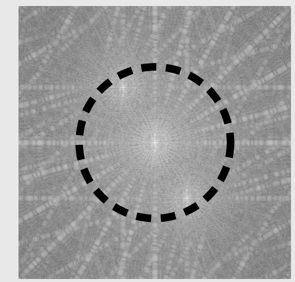
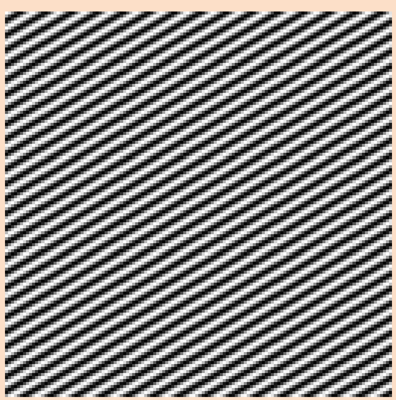
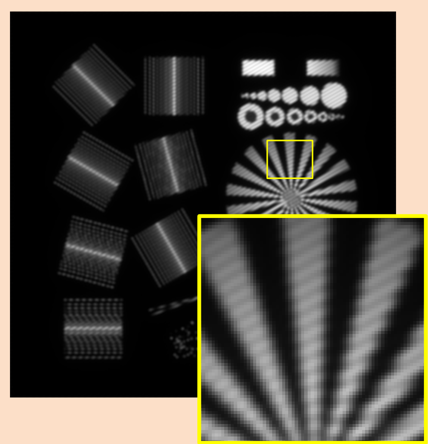


→ Need for additional measures to “close” the system of equations

Take 3 SIM images with different illumination phases

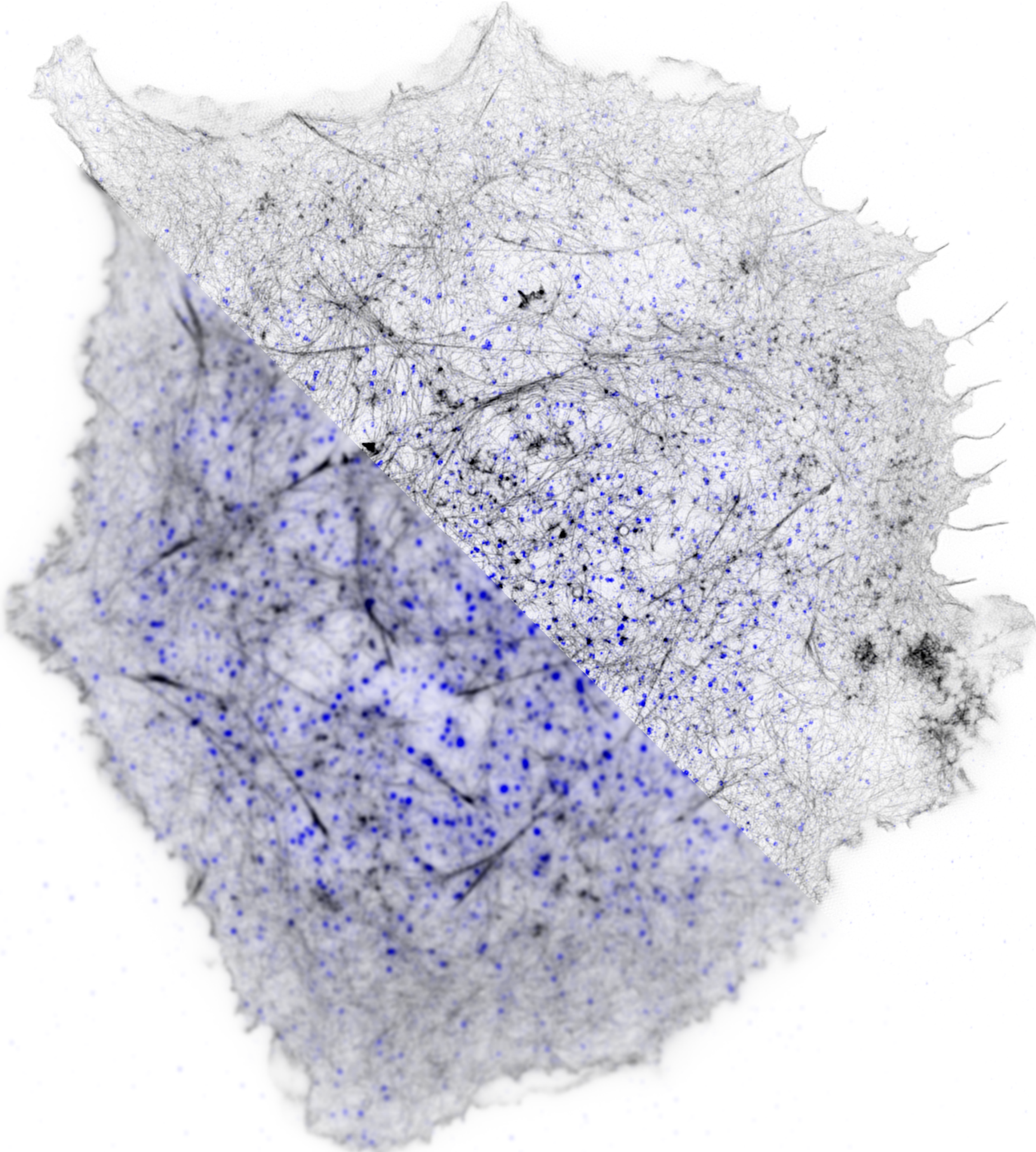




For $n = 0, 1, 2$

$$w_n(x) = 1 + \cos\left(fx + \frac{2n\pi}{3}\right)$$

Examples of SIM reconstructions



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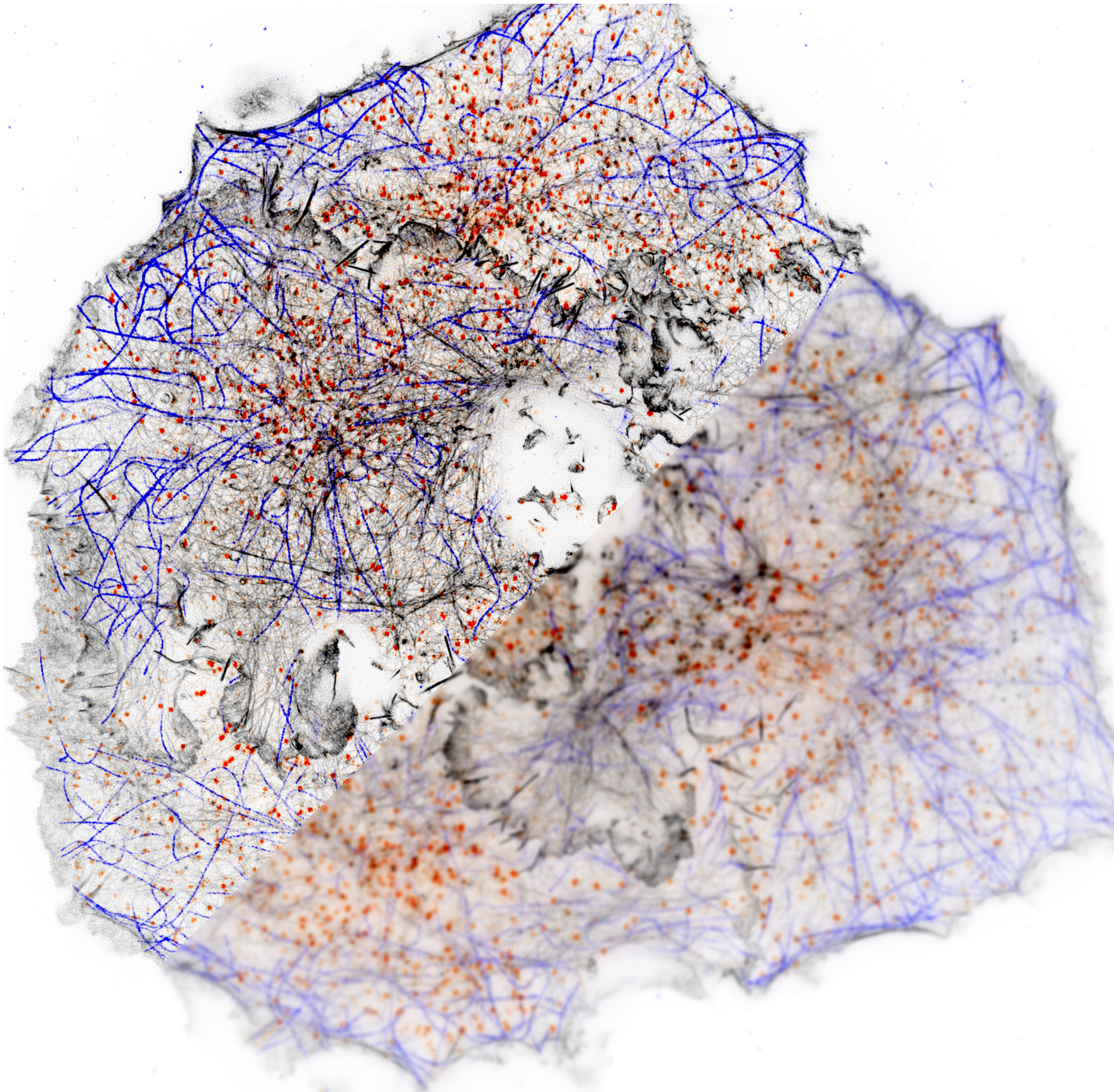

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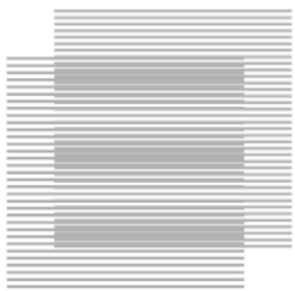

ADMM for SIM

GlobalBioIm

Variational Formulation for SIM Reconstruction

Image formation model

$$\boxed{\mathbf{y} = \mathbf{S_d H W x} + \mathbf{n},}$$

where
- $\mathbf{x} \in \mathbb{R}^N$ the 3D biological sample,
- $\mathbf{W} = \mathbf{diag}(\mathbf{w})$ with $\mathbf{w} \in \mathbb{R}^N_{\geqslant 0}$ an illumination pattern,
- $\mathbf{H} : \mathbb{R}^N \to \mathbb{R}^N$ the convolution operator with the PSF,
- $\mathbf{S_d} : \mathbb{R}^N \to \mathbb{R}^M$ a decimation operator with downsampling factor $\mathbf{d} \in \mathbb{N}^3_{\geqslant 0}$.

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Inverse problem: Variational formulation

$$\hat{\mathbf{x}} \in \left\{ \arg\min_{\mathbf{x}\in\mathbb{R}^N} \left(\sum_{p=1}^{P} \frac{1}{2} \|\mathbf{S}_{\mathbf{d}}\mathbf{H}\mathbf{W}_p\mathbf{x} - \mathbf{y}_p\|_2^2 + \mu\mathcal{R}(\mathbf{L}\mathbf{x}) + i_{\geqslant 0}(\mathbf{x}) \right) \right\},$$

with
- $\mathcal{R}(\mathbf{L}\cdot)$ a regularization term (e.g. Total variation, Hessian-Schatten-norm),
- The regularisation parameter $\mu > 0$,
- Nonnegativity constraint $i_{\geqslant 0}(\mathbf{x}) = \{0 \text{ if } \mathbf{x} \in \mathbb{R}^N_{\geqslant 0}, +\infty \text{ otherwise}\}$.

Alternating Direction Method of Multipliers (ADMM)

ADMM *[Boyd et al., 2011, Combette and Pesquet, 2011, Setzer et al, 2010]*

$$\mathcal{J}(\mathbf{x}) = \sum_{q=1}^{Q} \mathcal{F}_q(\mathbf{A}_q \mathbf{x}),$$

$\mathcal{F}_q : \mathbb{R}^{N_q} \to \mathbb{R}$ convex

$\mathbf{A}_q \in \mathbb{R}^{N_q \times N}$

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Principle

- Variable splitting

$$\operatorname*{arg\,min}_{\mathbf{x},\mathbf{u}_q} \sum_{q=1}^{Q} \mathcal{F}_q(\mathbf{u}_q) \quad \text{s.t.} \quad \mathbf{u}_q = \mathbf{A}_q \mathbf{x}$$

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- Alternating optimization on augmented Lagrangian formulation

$$\mathcal{L}(\mathbf{x}, \mathbf{u}_q, \mathbf{v}_q) = \sum_{q=1}^{Q} \mathcal{F}_q(\mathbf{u}_q) + \langle \mathbf{v}_q, \mathbf{A}_q \mathbf{x} - \mathbf{u}_q \rangle + \frac{\rho_q}{2} \|\mathbf{A}_q \mathbf{x} - \mathbf{u}_q\|_2^2$$

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- Proximal operator

$$\mathrm{prox}_f(\mathbf{z}) = \arg\min_{\mathbf{x}} \left(\frac{1}{2} \|\mathbf{x} - \mathbf{z}\|_2^2 + f(\mathbf{x}) \right)$$

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Require: $\mathbf{x}^0 \in \mathbb{R}^N$, $(\rho_q)_{q \in [1 \ldots Q]} \in \mathbb{R}^Q_{\geqslant 0}$

1: $\mathbf{u}_q^0 = \mathbf{A}\mathbf{x}^0$, $\forall q \in [1 \ldots Q]$
2: $\mathbf{v}_q^0 = \mathbf{u}_q^0$, $\forall q \in [1 \ldots Q]$
3: $k = 0$
4: **while** (not converged) **do**
5: $\quad \mathbf{u}_q^{k+1} = \mathrm{prox}_{\frac{1}{\rho_q}\mathcal{F}_q}\left(\mathbf{A}_q \mathbf{x}^k + \mathbf{v}_q^k / \rho_q\right)$, $\forall q \in [1 \ldots Q]$ ⟵ **Prox Step**
6: $\quad \mathbf{x}^{k+1} = \left(\sum_{q=1}^{Q} \rho_q \mathbf{A}_q^* \mathbf{A}_q\right)^{-1} \left(\rho_q \mathbf{A}_q^* \left(\mathbf{u}_q^{k+1} - \mathbf{v}_q^k / \rho_q\right)\right)$ ⟵ **Linear Step**
7: $\quad \mathbf{v}_q^{k+1} = \mathbf{v}_q^k + \rho_q(\mathbf{A}_q \mathbf{x}^{k+1} - \mathbf{u}_q^{k+1})$, $\forall q \in [1 \ldots Q]$
8: $\quad k = k + 1$
9: **end while**

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4: **while** (not converged) **do**
5: $\quad \mathbf{u}_q^{k+1} = \mathrm{prox}_{\frac{1}{\rho_q}\mathcal{F}_q}\left(\mathbf{A}_q\mathbf{x}^k + \mathbf{v}_q^k/\rho_q\right)$, $\forall q \in [1 \ldots Q]$ ⟵ **Prox Step**
6: $\quad \mathbf{x}^{k+1} = \left(\sum_{q=1}^{Q} \rho_q \mathbf{A}_q^* \mathbf{A}_q\right)^{-1} \left(\rho_q \mathbf{A}_q^* \left(\mathbf{u}_q^{k+1} - \mathbf{v}_q^k/\rho_q\right)\right)$ ⟵ **Linear Step**
7: $\quad \mathbf{v}_q^{k+1} = \mathbf{v}_q^k + \rho_q(\mathbf{A}_q\mathbf{x}^{k+1} - \mathbf{u}_q^{k+1})$, $\forall q \in [1 \ldots Q]$
8: $\quad k = k + 1$
9: **end while**

Which splitting should we adopt ?

Splitting Strategies for SIM

$$\hat{\mathbf{x}} \in \left\{ \arg\min_{\mathbf{x}\in\mathbb{R}^N} \left(\sum_{p=1}^{P} \frac{1}{2} \| \underbrace{\mathbf{S}_{\mathbf{d}}\mathbf{H}\mathbf{W}_p\mathbf{x}}_{?} - \mathbf{y}_p \|_2^2 + \mu \mathcal{R}(\underbrace{\mathbf{L}\mathbf{x}}_{\mathbf{u}_1}) + i_{\geqslant 0}(\underbrace{\mathbf{x}}_{\mathbf{u}_2}) \right) \right\}$$

	Prox Step	Linear Step
$\mathbf{A}_q = \mathbf{S}_{\mathbf{d}}\mathbf{H}\mathbf{W}_q$	$\mathrm{prox}_{\frac{\gamma}{2}\|\cdot - \mathbf{y}\|_2^2}(\mathbf{z})$ ► $(1+\gamma)^{-1}\mathbf{I}$ Component-wise division	$\left(\sum_{p=1}^{P} \mathbf{W}_p^\star \mathbf{H}^\star \mathbf{S}_{\mathbf{d}}^\star \mathbf{S}_{\mathbf{d}} \mathbf{H}\mathbf{W}_p + \mathbf{L}^\star\mathbf{L} + \mathbf{I}\right)^{-1}$ ► CG: 4xP (i)FFTs / Iter
$\mathbf{A}_q = \mathbf{H}\mathbf{W}_q$	$\mathrm{prox}_{\frac{\gamma}{2}\|\mathbf{S}_{\mathbf{d}}\cdot - \mathbf{y}\|_2^2}(\mathbf{z})$ ► $(\mathbf{I} + \gamma \mathbf{S}_{\mathbf{d}}^\star \mathbf{S}_{\mathbf{d}})^{-1}$ Component-wise division	$\left(\sum_{p=1}^{P} \mathbf{W}_p^\star \mathbf{H}^\star \mathbf{H}\mathbf{W}_p + \mathbf{L}^\star\mathbf{L} + \mathbf{I}\right)^{-1}$ ► CG: 2xP (i)FFTs / Iter
$\mathbf{A}_q = \mathbf{W}_q$	$\mathrm{prox}_{\frac{\gamma}{2}\|\mathbf{S}_{\mathbf{d}}\mathbf{H}\cdot - \mathbf{y}\|_2^2}(\mathbf{z})$ ► $(\mathbf{I} + \gamma \mathbf{H}^\star \mathbf{S}_{\mathbf{d}}^\star \mathbf{S}_{\mathbf{d}}\mathbf{H})^{-1}$???	$\left(\sum_{p=1}^{P} \mathbf{W}_p^\star \mathbf{W}_p + \mathbf{L}^\star\mathbf{L} + \mathbf{I}\right)^{-1}$ ► CG: 0 (i)FFTs / Iter

*CG: Conjugate-gradient

Proximal Operator of $\|\mathbf{S_d H} \cdot - \mathbf{y}\|_2^2$

Proposition 1 *The proximal operator of* $f = \frac{\gamma}{2}\|\mathbf{S_d H} \cdot - \mathbf{y}\|_2^2$ *can be efficiently computed at the expense of 2 (i)FFTs as*

$$\mathrm{prox}_f(\mathbf{z}) = \mathbf{F}^\star \left(\mathbf{I} - \gamma \boldsymbol{\Lambda}^\star \mathbf{P_d} \mathbf{D}^{-1} \mathbf{P}_\mathbf{d}^\star \boldsymbol{\Lambda}\right) \mathbf{F} \mathbf{r},$$

where

- $\mathbf{r} = \gamma \mathbf{H}^\star \mathbf{S}_\mathbf{d}^\star \mathbf{y} + \mathbf{z}$,
- $\mathbf{D} = \bar{d}\mathbf{I} + \gamma \mathbf{diag}\left(\mathbf{P}_\mathbf{d}^\star \boldsymbol{\Lambda}\boldsymbol{\Lambda}^\star \mathbf{1}_N\right)$, $\bar{d} = \prod_i d_i$,
- $\boldsymbol{\Lambda} = \mathbf{diag}\left(\mathbf{F}\mathbf{h}\right)$ *with* $\mathbf{h}$ *the kernel of* $\mathbf{H}$,
- $\mathbf{P_d}$ *is a periodization operator,*
- $\mathbf{F}$ *is the unitary DFT operator.*

Proximal Operator of $\|\mathbf{S_d H}\cdot - \mathbf{y}\|_2^2$

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<u>Quick Proof</u>

1) Woodbury matrix identity

$$(\mathbf{I} + \gamma \mathbf{H}^\star \mathbf{S_d^\star S_d H})^{-1} = \left(\mathbf{I} - \gamma \mathbf{H}^\star \mathbf{S_d^\star} (\mathbf{I} + \gamma \mathbf{S_d H H^\star S_d^\star})^{-1} \mathbf{S_d H}\right)$$

2) We can write $\mathbf{I} + \gamma \mathbf{S_d H H^\star S_d^\star}$ as a convolution

$$\begin{aligned}
\mathbf{I} + \gamma \mathbf{S_d H H^\star S_d^\star} &= \mathbf{I} + \gamma \mathbf{S_d F^\star} \boldsymbol{\Lambda}\boldsymbol{\Lambda}^\star \mathbf{F S_d^\star} \\
&= \mathbf{I} + \frac{\gamma}{\bar{d}} \mathbf{F^\star P_d^\star} \boldsymbol{\Lambda}\boldsymbol{\Lambda}^\star \mathbf{P_d F} \\
&= \mathbf{I} + \frac{\gamma}{\bar{d}} \mathbf{F^\star diag}\left(\mathbf{P_d^\star} \boldsymbol{\Lambda}\boldsymbol{\Lambda}^\star \mathbf{1}_N\right)\mathbf{F}
\end{aligned}$$

Proximal Operator of $\|\mathbf{S_d}\mathbf{H}\cdot - \mathbf{y}\|_2^2$

Proposition 1 *The proximal operator of* $f = \frac{\gamma}{2}\|\mathbf{S_d}\mathbf{H}\cdot - \mathbf{y}\|_2^2$ *can be efficiently computed at the expense of 2 (i)FFTs as*

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where
- $\mathbf{r} = \gamma\mathbf{H}^\star\mathbf{S_d^\star}\mathbf{y} + \mathbf{z}$,
- $\mathbf{D} = \bar{d}\mathbf{I} + \gamma\,\mathbf{diag}\left(\mathbf{P_d^\star}\mathbf{\Lambda}\mathbf{\Lambda}^\star\mathbf{1}_N\right)$, $\bar{d} = \prod_i d_i$,
- $\mathbf{\Lambda} = \mathbf{diag}\left(\mathbf{F}\mathbf{h}\right)$ *with* $\mathbf{h}$ *the kernel of* $\mathbf{H}$,
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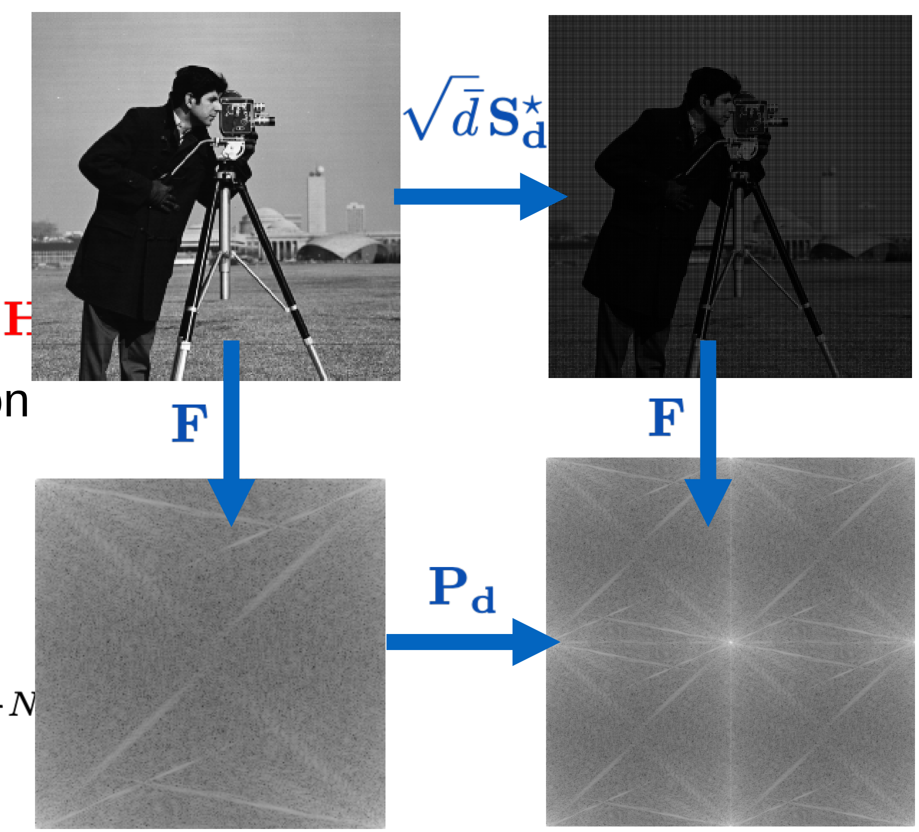

<u>**Quick Proof**</u>

1) Woodbury matrix identity

$$(\mathbf{I} + \gamma\mathbf{H}^\star\mathbf{S_d^\star}\mathbf{S_d}\mathbf{H})^{-1} = \left(\mathbf{I} - \gamma\mathbf{H}^\star\mathbf{S_d^\star}(\mathbf{I} + \gamma\mathbf{S_d}\mathbf{H}\right.$$

2) We can write $\mathbf{I} + \gamma\mathbf{S_d}\mathbf{H}\mathbf{H}^\star\mathbf{S_d^\star}$ as a convolution

$$\begin{aligned}\mathbf{I} + \gamma\mathbf{S_d}\mathbf{H}\mathbf{H}^\star\mathbf{S_d^\star} &= \mathbf{I} + \gamma\mathbf{S_d}\mathbf{F}^\star\mathbf{\Lambda}\mathbf{\Lambda}^\star\mathbf{F}\mathbf{S_d^\star}\\ &= \mathbf{I} + \frac{\gamma}{\bar{d}}\mathbf{F}^\star\mathbf{P_d^\star}\mathbf{\Lambda}\mathbf{\Lambda}^\star\mathbf{P_d}\mathbf{F}\\ &= \mathbf{I} + \frac{\gamma}{\bar{d}}\mathbf{F}^\star\mathbf{diag}\left(\mathbf{P_d^\star}\mathbf{\Lambda}\mathbf{\Lambda}^\star\mathbf{1}_N\right.\end{aligned}$$

Splitting Strategies for SIM

$$\hat{\mathbf{x}} \in \left\{ \arg\min_{\mathbf{x}\in\mathbb{R}^N} \left(\sum_{p=1}^{P} \frac{1}{2} \| \underbrace{\mathbf{S}_{\mathrm{d}}\mathbf{H}\mathbf{W}_p\mathbf{x}}_{?} - \mathbf{y}_p \|_2^2 + \mu \mathcal{R}(\underbrace{\mathbf{L}\mathbf{x}}_{\mathbf{u}_1}) + i_{\geqslant 0}(\underbrace{\mathbf{x}}_{\mathbf{u}_2}) \right) \right\}$$

	Prox Step	Linear Step
$\mathbf{A}_q = \mathbf{S}_{\mathrm{d}}\mathbf{H}\mathbf{W}_q$	$\mathrm{prox}_{\frac{\gamma}{2}\|\cdot - \mathbf{y}\|_2^2}(\mathbf{z})$ ► $(1+\gamma)^{-1}\mathbf{I}$ Component-wise division	$\left(\sum_{p=1}^{P} \mathbf{W}_p^\star \mathbf{H}^\star \mathbf{S}_{\mathrm{d}}^\star \mathbf{S}_{\mathrm{d}} \mathbf{H}\mathbf{W}_p + \mathbf{L}^\star\mathbf{L} + \mathbf{I}\right)^{-1}$ ► CG: 4xP (i)FFTs / Iter
$\mathbf{A}_q = \mathbf{H}\mathbf{W}_q$	$\mathrm{prox}_{\frac{\gamma}{2}\|\mathbf{S}_{\mathrm{d}}\cdot - \mathbf{y}\|_2^2}(\mathbf{z})$ ► $(\mathbf{I} + \gamma \mathbf{S}_{\mathrm{d}}^\star \mathbf{S}_{\mathrm{d}})^{-1}$ Component-wise division	$\left(\sum_{p=1}^{P} \mathbf{W}_p^\star \mathbf{H}^\star \mathbf{H}\mathbf{W}_p + \mathbf{L}^\star\mathbf{L} + \mathbf{I}\right)^{-1}$ ► CG: 2xP (i)FFTs / Iter
$\mathbf{A}_q = \mathbf{W}_q$	$\mathrm{prox}_{\frac{\gamma}{2}\|\mathbf{S}_{\mathrm{d}}\mathbf{H}\cdot - \mathbf{y}\|_2^2}(\mathbf{z})$ ► **2xP (i)FFTs**	$\left(\sum_{p=1}^{P} \mathbf{W}_p^\star \mathbf{W}_p + \mathbf{L}^\star\mathbf{L} + \mathbf{I}\right)^{-1}$ ► CG: 0 (i)FFTs / Iter

*CG: Conjugate-gradient

(Re)formulation of the Problem

Definition 1 *Let* $\mathbf{T} \in \mathbb{R}^{N \times N}$. *We say that* $\mathbf{T} \in \mathcal{N}_{\geqslant 0}$ *if and only if*

$$\mathbf{T}\mathbf{x} \in \mathbb{R}^N_{\geq 0}, \forall \mathbf{x} \in \mathbb{R}^N_{\geq 0}$$

$$\mathbf{T}\mathbf{x} \in \mathbb{R}^N \setminus \mathbb{R}^N_{\geq 0}, \forall \mathbf{x} \in \mathbb{R}^N \setminus \mathbb{R}^N_{\geq 0}$$

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Let $\mathbf{T} \in \mathcal{N}_{\geqslant 0}$, then we have that

$$\arg \min_{\mathbf{x}\in\mathbb{R}^N} \left(\mathcal{J}(\mathbf{x}) + i_{\geqslant 0}(\mathbf{x})\right) = \arg \min_{\mathbf{x}\in\mathbb{R}^N} \left(\mathcal{J}(\mathbf{x}) + i_{\geqslant 0}(\mathbf{T}\mathbf{x})\right)$$

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$$\hat{\mathbf{x}} \in \left\{ \arg \min_{\mathbf{x} \in \mathbb{R}^N} \left(\sum_{p=1}^{P} \frac{1}{2} \|\mathbf{S_d}\mathbf{H}\mathbf{W}_p\mathbf{x} - \mathbf{y}_p\|_2^2 + \mu\mathcal{R}(\mathbf{L}\mathbf{x}) + i_{\geqslant 0}(\mathbf{T}\mathbf{x}) \right) \right\}$$

where $\mathbf{T} = \left(\alpha\mathbf{I} - \mathbf{diag}\left(\sum_{p=1}^{P} \mathbf{w}_p \odot \mathbf{w}_p\right)\right)$ with $\alpha > \| \sum_{p=1}^{P} \mathbf{w}_p \odot \mathbf{w}_p\|_\infty$.

(Re)formulation of the Problem

Definition 1 *Let* $\mathbf{T} \in \mathbb{R}^{N \times N}$. *We say that* $\mathbf{T} \in \mathcal{N}_{\geqslant 0}$ *if and only if*

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$$\hat{\mathbf{x}} \in \left\{ \arg \min_{\mathbf{x} \in \mathbb{R}^N} \left(\sum_{p=1}^{P} \frac{1}{2} \| \mathbf{S_d H W}_p \mathbf{x} - \mathbf{y}_p \|_2^2 + \mu \mathcal{R}(\mathbf{Lx}) + i_{\geqslant 0}(\mathbf{Tx}) \right) \right\}$$

where $\mathbf{T} = \left(\alpha \mathbf{I} - \mathbf{diag} \left(\sum_{p=1}^{P} \mathbf{w}_p \odot \mathbf{w}_p \right) \right)$ with $\alpha > \| \sum_{p=1}^{P} \mathbf{w}_p \odot \mathbf{w}_p \|_\infty$.

The linear step in ADMM amounts to invert

$$\mathbf{B} = \sum_{p=1}^{P} \mathbf{W}_p^\star \mathbf{W}_p + \mathbf{L}^\star \mathbf{L} + \mathbf{T}^\star \mathbf{T} = \mathbf{L}^\star \mathbf{L} + \alpha \mathbf{I}$$

Splitting Strategies for SIM

$$\hat{\mathbf{x}} \in \left\{ \arg\min_{\mathbf{x}\in\mathbb{R}^N} \left(\sum_{p=1}^{P} \frac{1}{2} \| \underbrace{\mathbf{S_d H W}_p \mathbf{x}}_{?} - \mathbf{y}_p \|_2^2 + \mu \mathcal{R}(\underbrace{\mathbf{Lx}}_{\mathbf{u}_1}) + i_{\geqslant 0}(\underbrace{\mathbf{x}}_{\mathbf{u}_2}) \right) \right\}$$

	Prox Step	Linear Step
$\mathbf{A}_q = \mathbf{S_d H W}_q$	$\mathrm{prox}_{\frac{\gamma}{2}\|\cdot - \mathbf{y}\|_2^2}(\mathbf{z})$ ► $(1+\gamma)^{-1}\mathbf{I}$ Component-wise division	$\left(\sum_{p=1}^{P} \mathbf{W}_p^\star \mathbf{H}^\star \mathbf{S_d}^\star \mathbf{S_d H W}_p + \mathbf{L}^\star \mathbf{L} + \mathbf{I} \right)^{-1}$ ► CG: 4xP (i)FFTs / Iter
$\mathbf{A}_q = \mathbf{H W}_q$	$\mathrm{prox}_{\frac{\gamma}{2}\|\mathbf{S_d} \cdot - \mathbf{y}\|_2^2}(\mathbf{z})$ ► $(\mathbf{I} + \gamma \mathbf{S_d}^\star \mathbf{S_d})^{-1}$ Component-wise division	$\left(\sum_{p=1}^{P} \mathbf{W}_p^\star \mathbf{H}^\star \mathbf{H W}_p + \mathbf{L}^\star \mathbf{L} + \mathbf{I} \right)^{-1}$ ► CG: 2xP (i)FFTs / Iter
$\mathbf{A}_q = \mathbf{W}_q$	$\mathrm{prox}_{\frac{\gamma}{2}\|\mathbf{S_d H} \cdot - \mathbf{y}\|_2^2}(\mathbf{z})$ ► **2xP (i)FFTs**	$(\mathbf{L}^* \mathbf{L} + \alpha \mathbf{I})^{-1}$ ► **2 (i)FFTs**

*CG: Conjugate-gradient

Splitting Strategies for SIM

$$\hat{\mathbf{x}} \in \left\{ \arg\min_{\mathbf{x}\in\mathbb{R}^N} \left(\sum_{p=1}^{P} \frac{1}{2} \|\underbrace{\mathbf{S}_{\mathrm{d}}\mathbf{H}\mathbf{W}_p\mathbf{x}}_{?} - \mathbf{y}_p\|_2^2 + \mu\mathcal{R}(\underbrace{\mathbf{L}\mathbf{x}}_{\mathbf{u}_1}) + i_{\geqslant 0}(\underbrace{\mathbf{x}}_{\mathbf{u}_2}) \right) \right\}$$

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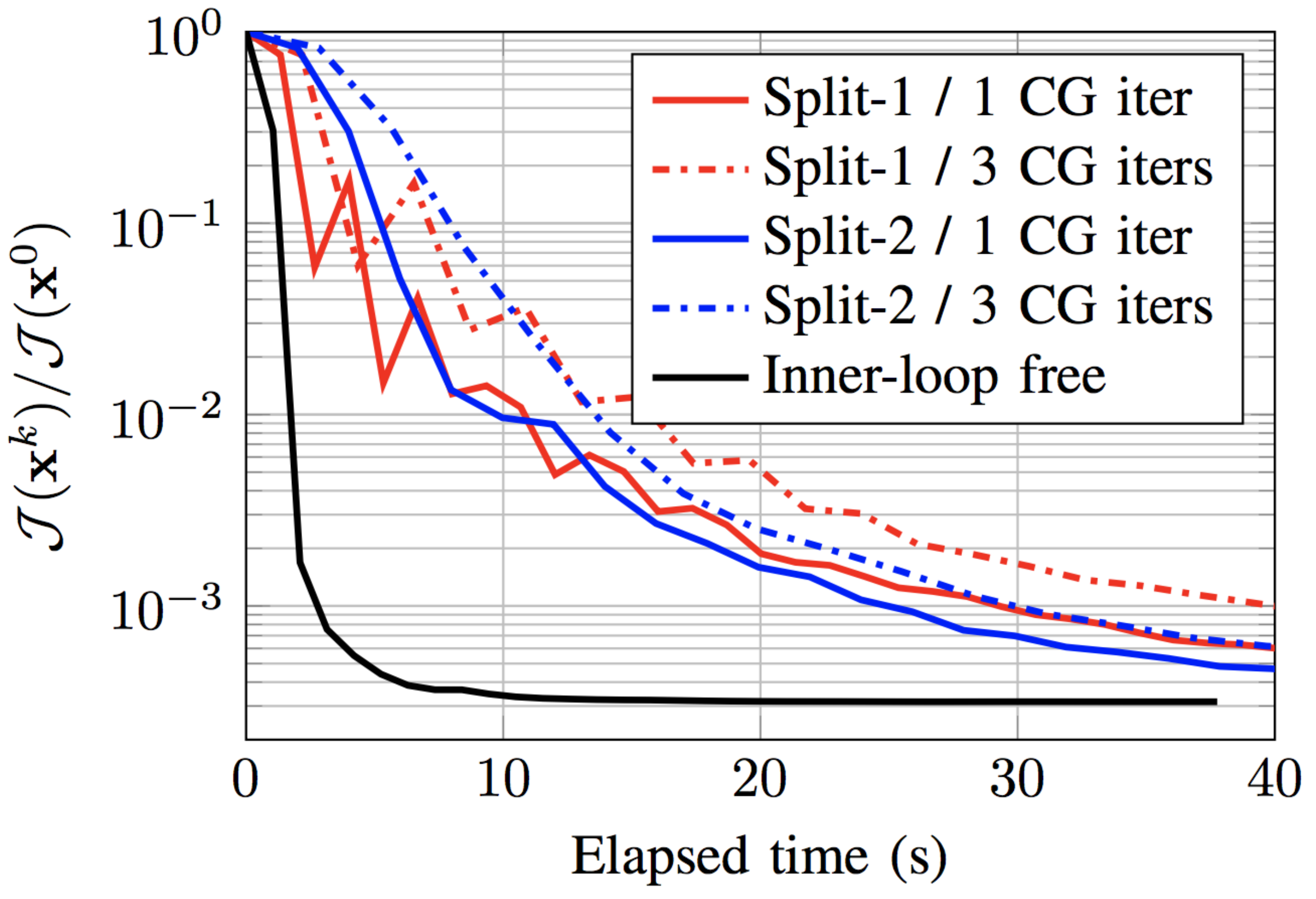
$\mathbf{A}_q = \mathbf{W}_q$	$\mathrm{prox}_{\frac{\gamma}{2}\|\mathbf{S}_{\mathrm{d}}\mathbf{H}\cdot - \mathbf{y}\|_2^2}(\mathbf{z})$ ► 2xP (i)FFTs	$(\mathbf{L}^*\mathbf{L} + \alpha\mathbf{I})^{-1}$

Inner-loop-free ADMM: 2x(P+1) (i)FFTs / iter !

Numerical Comparisons

- (512 x 512) size problem
- Lagrangian multipliers tuned empirically to get fastest convergence



Outline of the Talk

SIM Principle

ADMM for SIM

GlobalBioIm

GlobalBioIm Framework

Objectives

- Unification of inverse problems solving
- Decompose the problem in small reusable modules
- Capitalises on the commonalities between image formation models

GlobalBioIm Framework

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Main Features

- Generic implementation
- Automatic simplifications of composition of operators
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Three classes of modules

Cost Functions:
- *Evaluate*
- *Compute Gradient*
- *Compute Prox*
- *...*

Operators:
- *Apply forward*
- *Apply Jacobian*
- *...*

Optimizers
- *Minimize a cost*
- *...*

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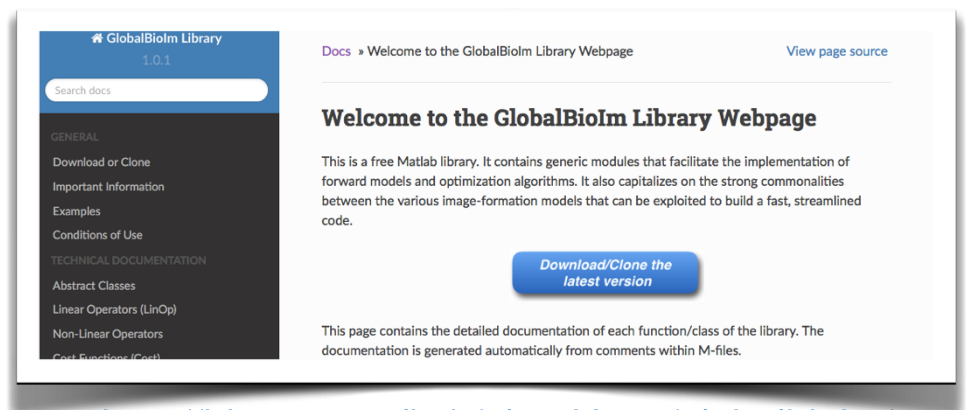

http://bigwww.epfl.ch/algorithms/globalbioim/

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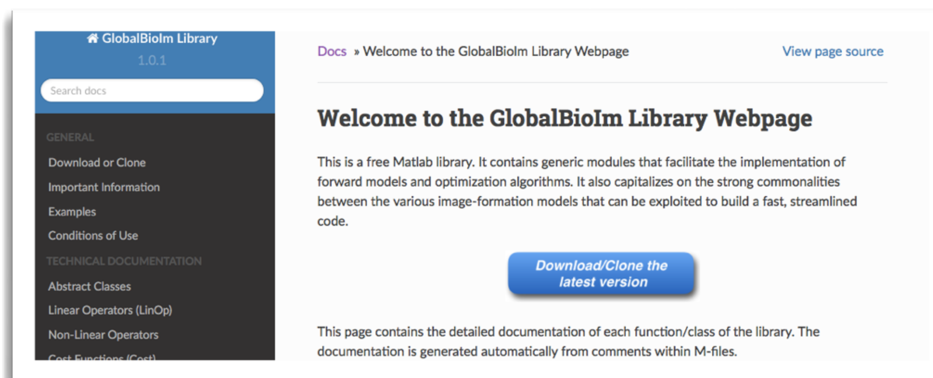

http://bigwww.epfl.ch/algorithms/globalbioim/

Python alternatives

- Deep Inverse
- Pyxu

ADMM for SIM Reconstruction with GlobalBioIm

Splitting 1

$$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^N} \left(\sum_{p=1}^{P} \frac{1}{2} \| \boxed{\mathbf{S}_{\mathrm{d}}\mathbf{H}\mathbf{W}_p\mathbf{x}} - \mathbf{y}_p \|_2^2 + \mu\mathcal{R}(\boxed{\mathbf{L}\mathbf{x}}) + i_{\geq 0}(\boxed{\mathbf{T}\mathbf{x}}) \right)$$

```
%- Data fidelity
H=LinOpConv(fftn(psf),1);
S=LinOpDownsample([szUp,H.sizeout(3)],[downFact,H.sizeout(3)]);
Fn={};Hn={};F0=[];
for i=1:size(patt,3)
    L2=CostL2([],y(:,:,i));
    Fn{i}=L2; Fn{i}.doPrecomputation=true;
   Hn{i}=S*H*LinOpDiag(H.sizein,patt(:,:,i));
end
%- Regularization
Opreg=LinOpHess(H.sizein,'circular',[1 2]);
Freg=CostMixNormSchatt1(Opreg.sizeout,1);
T=LinOpIdentity(H.sizein);
pos=CostNonNeg(H.sizein);

%- ADMM
FF=[Fn,{pos},{lamb*Freg}];
HH=[Hn,{OpD},{Opreg}];
rho=[ones(size(Fn))*rhoDTNN,rhoDTNN,rhoReg];
Opt=OptiADMM([],FF,HH,rho);
Opt.CvOp=TestCvgStepRelative(1e-5);
Opt.maxiter=500;
Opt.run(zeros(H.sizein));
```

ADMM for SIM Reconstruction with GlobalBioIm

Splitting 2

$$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^N} \left(\sum_{p=1}^{P} \frac{1}{2} \|\mathbf{S}_{\mathrm{d}} \boxed{\mathbf{H}\mathbf{W}_p\mathbf{x}} - \mathbf{y}_p\|_2^2 + \mu \mathcal{R}(\boxed{\mathbf{L}\mathbf{x}}) + i_{\geq 0}(\boxed{\mathbf{T}\mathbf{x}}) \right)$$

```
%- Data fidelity
H=LinOpConv(fftn(psf),1);
S=LinOpDownsample([szUp,H.sizeout(3)],[downFact,H.sizeout(3)]);
Fn={};Hn={};F0=[];
for i=1:size(patt,3)
    L2=CostL2([],y(:,:,i));
    Fn{i}=L2*S; Fn{i}.doPrecomputation=true;
   Hn{i}=H*LinOpDiag(H.sizein,patt(:,:,i));
end
%- Regularization
Opreg=LinOpHess(H.sizein,'circular',[1 2]);
Freg=CostMixNormSchatt1(Opreg.sizeout,1);
T=LinOpIdentity(H.sizein);
pos=CostNonNeg(H.sizein);

%- ADMM
FF=[Fn,{pos},{lamb*Freg}];
HH=[Hn,{OpD},{Opreg}];
rho=[ones(size(Fn))*rhoDTNN,rhoDTNN,rhoReg];
Opt=OptiADMM([],FF,HH,rho);
Opt.CvOp=TestCvgStepRelative(1e-5);
Opt.maxiter=500;
Opt.run(zeros(H.sizein));
```

ADMM for SIM Reconstruction with GlobalBioIm

Inner-loop free ADMM

$$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^N} \left(\sum_{p=1}^{P} \frac{1}{2} \|\mathbf{S}_{\mathrm{d}}\mathbf{H}\boxed{\mathbf{W}_p\mathbf{x}} - \mathbf{y}_p\|_2^2 + \mu\mathcal{R}(\boxed{\mathbf{L}\mathbf{x}}) + i_{\geq 0}(\boxed{\mathbf{T}\mathbf{x}}) \right)$$

```
%- Data fidelity
H=LinOpConv(fftn(psf),1);
S=LinOpDownsample([szUp,H.sizeout(3)],[downFact,H.sizeout(3)]);
Fn={};Hn={};F0=[];
for i=1:size(patt,3)
    L2=CostL2([],y(:,:,i));
    Fn{i}=L2*(S*H); Fn{i}.doPrecomputation=true;
   Hn{i}=LinOpDiag(H.sizein,patt(:,:,i));
end
%- Regularization
Opreg=LinOpHess(H.sizein,'circular',[1 2]);
Freg=CostMixNormSchatt1(Opreg.sizeout,1);
T=LinOpDiag(H.sizein,sqrt(alpha-sum(patt.^2,3)));
pos=CostNonNeg(H.sizein);

%- ADMM
FF=[Fn,{pos},{lamb*Freg}];
HH=[Hn,{OpD},{Opreg}];
rho=[ones(size(Fn))*rhoDTNN,rhoDTNN,rhoReg];
Opt=OptiADMM([],FF,HH,rho);
Opt.CvOp=TestCvgStepRelative(1e-5);
Opt.maxiter=500;
Opt.run(zeros(H.sizein));
```

Take home message

- ADMM is a very flexible method
- Offers several choices of splitting
- Can be tailored to each individual problem

Computational Super-Sectioning for Single-Slice Structured-Illumination Microscopy
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Thank you !